Chapter 6

VECTOR SPACES

EXERCISE 6.1 (Page 236)

1. Let V be the set of all infinite sequences in a field F with addition and scalar multiplication defined as below:

For $u = \{a_n\} = a_1, a_2, a_3, \cdots, a_n, \cdots \in V$

$v = \{b_n\} = b_1, b_2, b_3, \cdots, b_n, \cdots \in V,$

$u + v = \{a_n\} + \{b_n\} = a_1 + b_2, a_2 + b_2, \cdots, a_n + b_n, \cdots$

and $ku = k\{a_n\} = ka_1, ka_2, ka_3, \cdots, ka_n, \cdots$

where a_n, b_n and k are all in F, $n = 1, 2, 3, \cdots$.

Show that V is a vector space over F.

Sol. $u + v = \{a_n + b_n\} = \{a_n\} + \{b_n\},$

$ku = \{ka_n\} = k\{a_n\}.$

Let $w = \{c_n\} = c_1, c_2, \cdots, c_n, \cdots \in V$, then

$$(u + v) + w = \{a_n + b_n\} + \{c_n\}$$
$$= \{(a_n + b_n) + c_n\} = \{a_n + (b + c_n)\}$$
$$= \{a_n\} + \{b_n + c_n\} = u + (v + w)$$

Thus + is an associative binary operation on V.

The $\{0\}$ sequence $0, 0, 0, \cdots, 0, \cdots$ is the additive identity of V.

If $u = \{a_n\}$, then $-1 \cdot u = -a_1, -a_2, \cdots, -a_n, \cdots = \{-a_n\}$

so that $u + (-1 \cdot u) = 0, 0, 0, , 0, \cdots$

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Thus every $u \in V$ has additive inverse.

$$\begin{aligned} u + v &= a_1 + b_1, a_2 + b_2, \cdots, a_n + b_n, \cdots \\ &= b_1 + a_1, b_2 + a_2, \cdots, b_n + a_n, \cdots \\ &= \{b_n + a_n\} = \{b_n\} + \{a_n\} = v + u \end{aligned}$$

Thus + is commutative in V.

Hence V is an abelian group under addition.

Now we check the other axioms of a vector space for V.

Let $u = \{a_1, a_2, a_3, \cdots\}, v = \{b_1, b_2, b_3, \cdots\}$ be elements of V and $a, b \in F$.

$$\begin{aligned} \text{Then } a(u+v) &= a\{a_1 + b_1, a_2 + b_2, a_3 + b_3, \cdots\} \\ &= \{a(a_1 + b_1), a(a_2 + b_2), a(a_3 + b_3), \cdots\} \\ &= \{aa_1 + ab_1, aa_2 + ab_2, aa_3 + ab_3, \cdots\} \\ &= \{aa_1, aa_2, aa_3, \cdots\} + \{ab_1, ab_2, ab_3, \cdots\} \\ &= au + av. \\ (a+b)u &= (a+b)\{a_1, a_2, a_3, \cdots\} \\ &= \{(a+b)a_1, (a+b)a_2, (a+b)a_3, \cdots\} \\ &= \{aa_1 + ba_1, aa_2 + ba_2, aa_3 + ba_3, \cdots\} \\ &= \{aa_1, aa_2, aa_3, \cdots\} + \{ba_1, ba_2, ba_3, \cdots\} \\ &= au + au. \\ a(bu) &= a\{ba_1, ba_1, ba_3, \cdots\} \\ &= \{ab\,a_1, ab\,a_2, ab\,a_3, \cdots\} \\ &= (ab)\{a_1, a_2, a_3, \cdots\} \\ &= (ab)u. \\ 1u &= \{1a_1, 1a_2, 1a_3, \cdots\} \\ &= \{a_1, a_2, a_3, \cdots\} \\ &= u \end{aligned}$$

Hence V is a vector space over F.

2. Let V be the set of all ordered pairs of real numbers. Check whether V is a vector space over R with respect to the indicated operations. If not, state the axioms which fail to hold.

(i) $(a, b) + (c, d) = (a + c, b + d)$ and $k(a, b) = (ka, b)$

(i) Sol. (i) It is easy to check that V is an abelian group under as defined.

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(ii) If k is a real number, then

$$k[(a, b) + (c, d)] = k(a + c, b + d)$$
$$= (k(a + c), b + d)$$
$$= (ka + kc, b + d)$$
$$= (ka, b) + (kc, d)$$
$$= k(a, b) + k(c, d)$$

Thus axiom (ii) is satisfied.

(iii) Let k, l be real, numbers. Then

$$(k+l)(a, b) = ((k+l)a, b)$$

and $$k(a, b)+l(a, b) = (ka, b)+(la, b)$$
$$= (ka + la, b + b)$$
$$= ((k+l)a, 2b)$$

Thus $(k+l)(a, b) \neq k(a, b) + l(a, b)$ and so axiom (iii) doses not hold.

(iv) $$((kl)a, b) = (k(la), b) = k(la, b)$$
$$= k(l(a, b))$$

(v) $$1(a, b) = (1a, b) = (a, b)$$

The axiom (iii) fails to hold and so V is not a vector space.

2.(ii) $(a, b)+(c, d) = (a, b)$ and $k(a, b) = (ka, kb)$

Sol.

(i) V is not a group under + as defined above since any element (a, b) has no additive inverse.

(ii) $$k[(a, b) + (c, d)] = k(a, b) = (ka, kb)$$

and $$k(a, b) + k(c, d) = (ka, kb) + (kc, kd)$$
$$= (ka, kb)$$

Thus $k[(a, b) + (c, d)] = k(a, b) + k(c, d)$.

Therefore axiom (ii) is satisfied.

(iii) $$(k + l)(a, b) = (k + l)a, (k + l)b)$$
$$= (ka + la, kb + lb)$$
$$k(a, b) + l(a, b) = (kia, kb) + (la, lb) = (ka, kb)$$

Therefore, $(k + l)(a, b) \neq k(a, b) + l(a, b)$.

(iv) $k(l(a, b) = k(la, lb) = (kla, klb) = kl(a, b)$

(v) $1(1, b) = (1a, 1b) = (a, b)$.

V is not a vector space since axioms (i) and (iii) do not hold.

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2.(iii) $(a, b) + (c, d) = (a + c, b + d)$ and $k(a, b) = (k^2a, k^2b)$.

Sol.

(i) As in **2(i)**, it is easy to check that, V is an abelian group under addition as defined above.

(ii) $k[(a, b) + (c, d)] = k\big((a + c), (b + d)\big)$

$= \big(k^2(a + c), k^2(b + d)\big)$

$k(a, b) + k(c, d) = (k^2a, k^2b) + (k^2c, k^2d)$

$= \big(k^2(a + c), k^2(b + d)\big)$

and so $k((a, b) + (c, d)) = k(a, b) + k(c, d)$

(iii) $(k + l)(a, b) = \big((k + l)^2 a, (k + l)^2 b\big)$

$k(a, b) + l(a, b) = (k^2a, k^2b) + (l^2a, l^2b)$

$= \big((k^2 + l^2) a, (k^2 + l^2) b\big)$

Thus, $(k + l)(a, b) \neq k\big((a + c), (b + d)\big)$

(iv) $(kl)(a, b) = (k^2l^2a, k^2l^2b)$

$= k(l^2a, l^2b)$

$= k(l(a, b)$

(v) $1(a, b) = (1^2a, 1^2 b) = (a, b)$

Thus axiom (iii) is not satisfied and so V is not a vector space.

2.(iv) $(a, b) + (c, d) = (a + c, b + d)$ and $k(a, b) = (ka, 0)$.

Sol. (i) V is an abelian group under + as defined.

(ii) $k((a, b) + (c, d)) = k\big((a + c, b + d)\big) = \big(k(a + c), 0\big)$

$= (ka + kc, 0) = (ka, 0) + (kc, 0)$

$= k(a, b) + k(c, d)$.

(iii) $(k + l)(a, b) = \big((k + l) a, 0\big) = (ka + la, 0)$

$= (ka, 0) + (la, 0) = k(a, b) + l(a, b)$

(iv) $(kl)(a, b) = (kla, 0)$

$k\big(l(a, b)\big) = k(la, 0) = (kla, 0)$

Therefore, $k\big((a, b)\big) = (kl)(a, b)$

(v) $1(a, b) = (1 \cdot a, 0) = (a, 0) \neq (a, b)$.

Thus axiom (v) is not satisfied and so V is not a vector space.

3. Check whether each of the following is a real vector space

(i) The set $C[a, b]$ of all continuous real-valued functions defined on $[a, b]$ with the usual operations on functions as defined in **Example 7.**

Sol. It is known from Calculus that if $f, g \in C[a, b]$ and k is a real number then

$f \pm g \in C[a, b]$ and $kf \in C[a, b]$

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The 0 function, being constant, is continuous and so $0 \in C[a, b]$. As in Example 16, + is an associative binary operation on $C[a, b]$. All the group axioms are satisfied in $C[a, b]$ under +. Thus $C[a, b]$ is an abelian group under +.

We note that the other axioms of a vector space for $C[a, b]$ are satisfied since they are satisfied in its superset of all real functions defined on $[a, b]$. **(Example 7)**

Thus $C[a, b]$ is a vector space.

3.(ii) The set of all functions $f \in C[a, b]$ such that $f(a) = f(b)$

Sol. Since $0(a) = 0 = 0(b)$, the 0 function is in the given set V. If $f \in V$, then $f(a) = f(b)$ and so $-f(a) = -f(b)$ implying that $-f \in V$. Let $f, g \in V$ then

$$f(a) = f(b) \quad \text{and} \quad g(a) = g(b)$$

Now $(f + g)(a) = f(a) + g(a)$

$$= f(b) + g(b) = (f + g)(b)$$

and so $f + g \in V$

V satisfies all the axioms of an abelian group under addition.

The remaining axioms of a vector space are also satisfied for V since V is a subset of $C[a, b]$ and $C[a, b]$ is a vector space. Hence V is a real vector space.

Alternative Method:

We show that V is a subspace of $C[a, b]$. Let $f, g \in V$ and $k, l \in R$

$$(kf + lg)(a) = kf(a) + lg(a)$$

$$= kf(b) + lg(b)$$

$$= (kf + lg)(b)$$

and so $kf + lg \in V$. By (6.5), V is a subspace of $C[a, b]$.

3.(iii) The set of all solutions of the differential equation

$$\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = 0$$

Sol. Let V be the set of all solutions of the given differential equation. Then V is a subset of all real functions defined on R. We show that V is a subspace of the vector space of Example 7.

Let $f, g \in V$. Then

$$\frac{d^2f}{dx^2} - 5\frac{df}{dx} + 6f = 0, \text{ i.e., } f'' - 5f' + 6f = 0$$

and $$\frac{d^2g}{dx^2} - 5\frac{dg}{dx} + 6g = 0, \text{ i.e., } g'' - 5g' + 6g = 0$$

We shall show that $af + bg \in V$, whenever $f, g \in V, a, b \in R$.

Now $(af + bg)'' - 5(af + bg)' + 6(af + bg)$

$= af'' + bg'' - 5af' - 5bg' + 6af + 6bg$

$= a(f'' - 5f' + 6f) + b(g'' - 5g' + 6g)$

$= a \cdot 0 + b \cdot 0 = 0$

Thus $af + bg \in V$ and therefore, by (6.5), V is a subspace of the vector space of all real functions defined on R.

3.(iv) The set of all 2×2 real matrices of the form $\begin{bmatrix} a & 1 \\ 1 & b \end{bmatrix}$

Sol. The matrix $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ is not in the given set and so it has no additive identity. Thus it is not a group under matrix addition. The first axiom for a vector space is not satisfied and so the given set is not a vector space.

4. Check whether each of the following subsets is a subspace of the indicted vector space:

(i) Q the set of rational numbers in R.

Sol. If $p, q \in Q$ and $a, b \in R$ then $ap + bq$ need not be a rational number. Therefore, by (6.5), Q is not a subspace of R.

(ii) All 2×2 nonsingular real matrices in M_{22}.

Sol. Since the given set consists of 2×2 nonsingular real matrices, the zero matrix $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ is not in the set. Thus the set is not a group and so it is not a subspace of M_{22}.

(iii) The set $B[a, b]$ of all bounded real functions defined on $[a, b]$ in the space of all real functions defined on $[a, b]$.

Sol. Let $k, l \in R$ and $f, g \in B[a, b]$.

Then $kf + lg$ is a bounded real function defined on $[a, b]$. Thus $kf + lg \in B[a, b]$ and therefore, by (6.5), $B[a, b]$ is a subspace of the space of all real functions defined on $[a, b]$.

5. Show that the union of two subspaces of a vector space need not be a subspace. Let X and Y be subspaces of a vector space V. Prove that $X \cup Y$ is s subspace of V if and only if either $X \subset Y$ or $Y \subset X$.

Sol. In the Euclidean space R^3, let

$$X = \{(x_1, 0, 0) : x_1 \in R\}, \quad Y = \{(0, x_2, 0) : x_2 \in R\}$$

Then it is easy to see that both X, Y are subspaces of R^3. We shall show that $X \cup Y$ is not a subspace of R^3.

The vectors $x = (x_1, 0, 0)$ and $y = (0, x_2, 0)$ are in $X \cup Y$. But $x + y = (x_1, 0, 0) + (0, x_2, 0) = (x_1, x_2, 0) \notin X \cup Y$.

Thus condition (i) of Theorem 6.4 is not satisfied and so $X \cup Y$ is not subspace of R^3.

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Suppose $X \cup Y$ is a subspace of V and let $X \not\subset Y$ and $Y \not\subset X$. Then there exists $x \in X$ such that $x \notin Y$ and there is at least one $y \in Y$ such that $y \notin X$.

Now $x, y \in X \cup Y$ and since $X \cup Y$ is a vector subspace, $(x + y) \in X \cup Y$. If $x + y \in X$, then $y = (x + y) - x \in X$ (since X is a vector subspace) contrary to our supposition.

Similarly, if $(x + y) \in Y$, then $x = (x + y) - y \in Y$, contradicting our construction. Thus $X \not\subset Y$ and $Y \not\subset X$ is impossible and so either $X \subset Y$ or $Y \subset X$.

Conversely, if $X \subset Y$, then $X \cup Y = Y$ which is a subspace of V. Similarly, if $Y \subset X$, then $X \cup Y = X$, which is a subspace of V.

6. Which of the following are subspaces of R^3?

ii) $W = \{(x, y, z) : x + y + z = 0\}$

Sol. $W = \{(x, y, z) \; x + y + z = 0\}$

Let $u, v \in W$ where

$u = (x_1, y_1, z_1)$ and $x_1 + y_1 + z_1 = 0$,

$v = (x_2, y_2, z_2)$ and $x_2 + y_2 + z_2 = 0$

Let $a, b \in R$.

Now, $au + bv = (ax_1, ay_1, az_1) + (bx_2, by_2, bz_2)$

$= (ax_1 + bx_2, ay_1 + by_2, az_1 + bz_2)$

and $ax_1 + bx_2 + ay_1 + by_2 + az_1 + bz_2$

$= a(x_1 + y_1 + z_1) + b(z_2 + y_2 + z_2)$

$= a0 + b0 = 0 + 0 = 0$

Thus $au + bv \in W$ hence W is a subspace of R^3.

6.(ii) $W = \{(x, y, z) : x \geq 0\}$

Sol. Let $u = (x, y, z) : x \geq 0$ so that $u \in W$. If k is a negative real number, then ku is either 0 or a negative real number. Thus $ku \notin W$. Therefore, W is not closed under scalar multiplication. Hence W is not a subspace of R^3.

(iii) $W = \{(x, y, z): x^2 + y^2 + z^2 \leq 1\}$

Sol. Here $W = \{(x, y, z): x^2 + y^2 + z^2 \leq 1\}$

Let $u = (1, 0, 0)$, $v = (0, 1, 0)$

Since $1^2 + 0^2 + 0^2 \leq 1$, $0^2 + 1^2 + 0^2 \leq 1$, therefore $u, v \in W$.

Let $a = b = 1$. Then $a, b \in R$.

Now $1u + 1v = u + v = (1, 1, 0)$.

But $1^2 + 1^2 + 0^2 = 2 > 1$, therefore $(1, 1, 0) \notin W$.

Thus is W is not a subspace of R^3

6.(iv) $W = \{(x, y, z): x, y, z \text{ are rationals}\}$

Sol. Let $u = (x, y, z) \in W$. Then

$ku = k(x, y, z) = (kx, ky, kz)$.

Now if k is an irrational number then kx, ky, kz are irrationals and so $ku \notin W$. Thus W is not closed under scalar multiplication and so W is not a subspace of R^3.

6.(v) $W = \{(x, 0, z): x, y, z \in R\}$

Sol. • Let $u = (x_1, 0, z_1)$ $v = (x_2, 0, z_2)$ be any elements of W. Then for all real numbers k, l, we have

$$ku + lv = k(x_1, 0, z_1) + l(x_2, 0, z_2)$$
$$= (kx_1 + lx_2, 0, kz_1 + lz_2)$$

which also belongs to W. Therefore by (6.5), W is a subspace of R^3.

6.(vi) $W = \{(x, y, z): x, y, z \in R, y^2 = x + z\}$

Sol. Let $u = (4, 3, 5)$, $v = (10, 5, 15)$

Then $u, v \in W$. Now $u + v = (4, 3, 5) + (10, 5, 15)$

$$= (14, 8, 20) \notin W$$

because $8^2 = 64 \neq 14 + 20 = 34$

Thus W is not a subspace of R^3

6.(vii) $W = \{(x, y, z): x, y, z \in R, 2x + 3y - 4z = 0\}$

Sol. Let $u = (x_1, y_1, z_1)$ and $v = (x_2, y_2, z_2)$ be any two elements of W. Then

$$2x_1 + 3y_1 - 4z_1 = 0$$
$$2x_2 + 3y_2 - 4z_1 = 0$$

For $a, b \in R$, we have

$$au + bv = (ax_1, ay_1, az_1) + (bx_2, by_2, bz_2)$$
$$= (ax_1 + bx_2, ay_1 + by_2, az_1 + bz_2)$$

Substituting into $2x + 3y - 4z$, we have

$$2(ax_1 + bx_2) + 3(ay_1 + by_2) - 4(az_1 + bz_2)$$
$$= a(2x_1 + 3y_1 - 4z_1) + b(2x_2 - 3y_2 + 4z_2) = a \cdot 0 + b \cdot 0 = 0$$

Thus $au + bv \in W$ and so W is a subspace of R^3.

7. Let V be the vector space of all real-valued functions defined on R. State which of the following are subspaces of V.

(i) The set of all even functions.

Sol. $W = \{f \in V : f \text{ is an even function}\}$

Let $f, g \in W$. Then for all $x \in R$,

$f(-x) = f(x)$, $g(-x) = g(x)$.

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Let $a, b \in R$

Then $(af + bg)(-x) = af(-x) + bg(-x) = af(x) + bg(x)$
$= (af + bg)(x)$

Thus $af + bg \in W$.

Hence W is a subspace of V.

7.(ii) The set of all differentiable functions

Sol. $W = \{ f \in V : f$ is a differentiable function $\}$

Let $f, g \in W$ Then f and g are differentiable, i.e., f' and g' exist. Let $a, b \in R$

Then $(af + bg)' = af' + bg'$

Thus $af + bg$ is differentiable. So $af + bg \in W$.

Hence W is subspace of V.

7.(iii) The set $W = \{ f : f(x) = kf(-x), k \in R$ fixed$\}$

Sol. Let $f, g \in W$. Then $f(x) = kf(-x)$ and $g(x) = kg(-x)$.

If $\alpha, \beta \in R$ then

$$(\alpha f + \beta g)(x) = \alpha f(x) + \beta g(x)$$
$$= \alpha k f(-x) + \beta k g(-x)$$
$$= k(\alpha f + \beta g)(-x)$$

Therefore,

$\alpha f + \beta f \in W$ and so W is a subspace of V

7.(iv) The set $W = \left\{ f \in V : \int_0^1 f(x)\,dx = 0 \right\}$.

Sol. Let $f, g \in W$. Then

$\int_0^1 f = 0$ and $\int_0^1 g = 0$. If $\alpha, \beta \in R$, then

$$\int_0^1 \alpha f + \beta g = \alpha \int_0^1 f + \beta \int_0^1 g = \alpha 0 + \beta 0 = 0$$

Therefore, $\alpha f + \beta g \in W$ and so W is a subspace of V.

8. Let V be the vector space of all real polynomials of degree $\leq n$ together with the zero polynomial. Determine whether or not W is a subspace of V, where W consists of the zero polynomial and all polynomials

(i) with integral coefficients and of degree $\leq n$.

Sol. Let $p(x) = a_0 + a_1 x + \cdots + a_n x^n$ where $a_0, a_1, \cdots, a_n \in Z$.

Then $p(x) \in W$. If α is a fraction or an irrational number then $\alpha p(x) = a_0 \alpha + a_1 \alpha x + \cdots a_n \alpha x^n \notin W$ since $a_0 \alpha, a_1 \alpha, \cdots, a_n \alpha$ are not integers.

Thus W is not closed under scalar multiplication and so W not a subspace of V.

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8.(ii) of degree ≤ 3.

Sol. $W = \{ p(x) : \deg p(x) \leq 3 \}$

If $p_1(x), p_2(x) \in W$, then $\deg p_1(x) \leq 3$

$\deg p_2(x) \leq 3$. If $a, b \in R$, then

$\deg (ap_1(x) + bp_2(x)) \leq 3$

Hence $ap_1(x) + bp_2(x) \in W$.

Thus W is a subspace of V.

8.(iii) With only even powers of x and of degree $\leq n$

Sol. $W = \{ p(x) : p(x) = b_0 + b_1x^2 + b_2x^4 + \cdots + b_{2k}x^{2k} \}$. $2k \leq n$

Suppose $u, v \in W$

Let $u = c_0 + c_1x^2 + c_2x^4 + \cdots + c_{2k}x^{2k}$,

$v = d_0 + d_1x^2 + d_2x^4 + \cdots + d_{2k}x^{2k}$

where some coefficients may be zero.

If $a, b \in$ R, then

$$au + bv = (ac_0 + bd_0) + (ac_1 + bd_1)x^2 + (ac_2 + bd_2)x^4 + \cdots + (ac_{2k} + bd_{2k})x^{2k} \in W.$$

Thus W is a subspace of V.

9. Express the vector $(2, -5, 3)$ in R^3 as a linear combination of the vectors $(1, -3, 2)$, $(2, -4, -1)$ and $(1, -5, 7)$.

Sol. Let $(2, -5, 3) = a(1, -3, 2) + b(2, -4, -1) + c(1, -5, 7)$

$= (a + 2b + c, -3a - 4b - 5c, 2a - b + 7c)$ (1)

Therefore, $a + 2b + c = 2$ (2)

$-3a - 4b - 5c = -5$ (3)

$2a - b + 7c = 3$ (4)

Multiply (2) by 3 and add the result to (3). Then

$2b - 2c = 1$ i.e., $b - c = \frac{1}{2}$ (5).

Multiply (2) by 2 and subtract the result from (4) to have

$-5b + 5c = -1$

i.e. $b - c = \frac{1}{2}$ (6)

Equation (5) and (6) are inconsistent.

Thus it is not possible to find a, b, c so as to satisfy (1).

Hence $(2, -5, 3)$ is not expressible as a linear combination of the given vectors.

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10. For what value of k will the vector $(1, -2, k)$ in R^3 be a linear combination of the vectors $(3, 0, -2)$ and $(2, -1, -5)$?

Sol. Suppose $(1, -2, k) = a(3, 0, -2) + b(2, -1, -5)$

$= (3a + 2b, -b, -2a - 5b)$, where $a, b \in R$

Therefore, $3a + 2b = 1$ (1)

$-b = -2$ (2)

$-2a - 5b = k$ (3)

From (1) and (2), we have $a = -1$, $b = 2$

From (3), we get $-2(-1) - 5(2) = k$ or $2 - 10 = k$

i.e., $k = -8$.

11. Let U and W be the subspaces of R^3 defined by

$U = \{(x, y, z) : x = y = z\}$ and

$W = \{(0, y, z) : y, z \in R\}$.

Show that $R^3 = U \oplus W$

Sol. $U = \{(x, y, z) : x = y = z\}$, $W = \{(0, y, z) : y, z \in R\}$

Let $(x_1, y_1, z_1) \in R^3$.

Then $(x_1, y_1, z_1) = (x_1, x_1, x_1) + (0, y_1 - x_1, z_1 - x_1) \in U + W$

Thus $R^3 \subset U + W$

Conversely, if $u \in U$, $w \in W$, then $u = (x_1, y_1, z_1)$, $w = (0, y_2, z_2)$

and $u + w = (x_1, y_1 + y_2, z_1 + z_2) \in R^3$

Thus $U + W \subset R^3$ and so $R^3 = U + W$.

Let $\alpha \in U \cap W$ Then $\alpha = (x, x, x)$ and $\alpha = (0, y, z)$

Now $(x, x, x) = (0, y, z)$ implies $x = 0$, $y = 0$, $z = 0$

Thus $\alpha = (0, 0, 0)$.

Hence $U \cap W = \{(0, 0, 0)\}$, showing that $R^3 = U \oplus W$

12. Show that each of the following sets of vectors generates R^3

(i) $\{(1, 2, 3), (0, 1, 2), (0, 0, 1)\}$.

Sol. Let $(x, y, z) \in R^3$

Suppose $(x, y, z) = a(1, 2, 3) + b(0, 1, 2) + c(0, 0, 1)$

$= (a, 2a + b, 3a + 2b + c)$, $a, b, c \in R$

Hence $x = a$ (1)

$y = 2a + b$ (2)

$z = 3a + 2b + c$ (3)

From (2) and (1), we get

$b = y - 2a = y - 2x$

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and $c = z - 3a - 2b$ from (3)

$= z - 3x - 2(y - 2x)$

$= z - 3x - 2y + 4x = x + x - 2y$

Thus we can express any element (x, y, z) of R^3 as a linear combination of the vectors $(1, 2, 3)$, $(0, 1, 2)$ and $(0, 0, 1)$.

Hence the given vectors generate R^3

12.(ii) $\{(1, 1, 1), (0, 1, 1), (0, 1, -1)\}$

Sol. Let $(x, y, z) = a(1, 1, 1) + b(0, 1, 1) + c(0, 1, -1)$

$= (a, a + b + c, a + b - c)$ where $a, b, c \in R$

Therefore, $a = x$ (1)

$a + b + c = y$ (2)

$a + b - c = z$ (3)

Adding (2) and (3), we have

$2a + 2b = y + z$

$a + b = \frac{y+z}{2}$

or $x + b = \frac{y+z}{2}$, using (1)

i.e., $b = \frac{y+z}{2} - x$

$= \frac{y+z-2x}{2}$

Now from (2), we get

$c = y - a - b$

$= y - x - \frac{y+z-2x}{2}$

$= \frac{2y - 2x - y - z + 2x}{2} = \frac{y-z}{2}$

Thus any element $(x, y, z) \in R^3$ can be expressed as a linear combination of $(1, 1, 1)$, $(0, 1, 1)$ and $(0, 1, -1)$. Hence these vectors generate R^3.

13. Determine whether the set $S = \{(1, 1, 2), (1, 0, 1), (2, 1, 3)\}$ spans R^3.

Sol. Let $u = (x, y, z)$ be an arbitrary element of R^3. The set S will span R^3 if u is a linear combination of members of S. Suppose

$(x, y, z) = a(1, 1, 2) + b(1, 0, 1) + c(2, 1, 3)$, where $a, b, c \in R$.

$= (a + b + 2c, a + c, 2a + b + 3c)$

Therefore,

$x = a + b + 2c$ (1)

$y = a + c$ (2)

$z = 2a + b + 3c$ (3)

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From (2), $c = y - a$. Substituting into (1) and (3), we have

$$x = a + b + 2y - 2a = b - a + 2y$$
$$z = 2a + b + 3y - 3a = b - a + 3y$$

These equations gives no solution for a and b. Thus (x, y, z) cannot be expressed as a linear combination of members of S and so S does not span R^3.

14. Show that the yz-plane

$$W = \{(0, y, z) : y, z \in R\}$$ is spanned by

(i) $(0, 1, 1)$ and $(0, 2, -1)$

Sol. $W = \{(0, y, z) | y, z \in R\}$

Let $(0, y, z) = a(0, 1, 1) + b(0, 2, -1)$, for $a, b \in R$

$= (0, a + 2b, a - b)$

Therefore, $a + 2b = y$ (1)

$a - b = z$ (2)

Subtracting (2) from (1), we get

$$3b = y - z \quad \text{i.e.} \quad b = \frac{y - z}{3}$$

$$a = z + b \quad \text{from (2)}$$

$$= z + \frac{y - z}{3} = \frac{2z + y - z}{3} = \frac{2z + y}{3}$$

Thus W is generated by $(0, 1, 1)$ and $(0, 2, -1)$

14.(ii) $(0, 1, 2)$, $(0, 2, 3)$ and $(0, 3, 1)$

Sol. Let $(0, y, z) = a(0, 1, 2) + b(0, 2, 3) + c(0, 3, 1)$, for $a, b, c \in R$

$= (0, a + 2b + 3c, 2a + 3b + c)$

Therefore, $a + 2b + 3c = y$

$2a + 3b + c = z$

Suppose we choose $c = 0$

Then $a + 2b = y$ (1)

$2a + 3b = z$ (2)

Multiply (1) by 3 and (2) by 2 to have

$3a + 6b = 3y$ 3)

$4a + 6b = 2z$ (4)

Subtracting (4) from (3), we obtain

$-a = 3y - 2z$

Thus $a = 2z - 3y$

Now $2b = y - a = y - 2z + 3y = 4y - 2z$ i.e. $b = 2y - z$

Hence W is generated by the given vectors

15. Find an equation (or equations) of the subspace W of R^3 spanned by each of the following sets of vectors:

(i) $\{(1, -3, 5), (-2, 6, -10)\}$

Sol. A vector (x, y, z) is in the span of the given set if for $a, b \in R$

$$(x, y, z) = a(1, -3, 5) + b(-2, 6, -10)$$
$$= (a - 2b, -3a + 6b, 5a - 10b)$$

i.e. the system of equations

$$x = a - 2b = t$$
$$y = -3a + 6b = -3t$$
$$z = 5a - 10b = 5t$$

is consistent. But these are parametric equation of a straight line whose direction vector is $(1, -3, 5)$. Required equations of the subspace are

$$x = t, \quad y = -3t, \quad z = 5t.$$

15.(ii) $\{(1, -3, 2), (-2, 0, 3)\}$

Sol. Let (x, y, z) be any vector in the span of the given set

Then for $a, b \in R$

$$(x, y, z) = a(1, -3, 2) + b(-2, 0, 3)$$
$$= (a - 2b, -3a, 2a + 3b)$$

i.e.,
$$x = a - 2b$$
$$y = -3a$$
$$z = 2a + 3b$$

is a consistent system of equations in the unknowns a and b. Or the equivalent system:

$$x = a - 2b$$
$$y = -3a$$
$$9x + 7y + 6z = 0$$

is consistent and it is the case if and only if

$$9x + 7y + 6z = 0$$

This is an equation of a plane representing the subspace spanned by the given set

15.(iii) $\{(1, -2, 1), (-2, 0, 3), (3, -2, -2)\}$

Sol. A vector (x, y, z) is in the span of the given set of vectors if it is a linear combination of the vectors of the set i.e.,

$$(x, y, z) = a(1, -2, 1) + b(-2, 0, 3) + c(3, -2, -2); a, b, c \in R$$
$$= (a - 2b + 3c, -2a - 2c, a + 3b - 2c)$$

That is a system of equations in the unknowns a, b, c :

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Hence
$$\begin{aligned} a - 2b + 3c &= x \\ -2a \quad -2c &= y \\ a + 3b - 2c &= z \end{aligned}$$
is consistent.

Or the equivalent system
$$\begin{aligned} a - 2b + 3c &= x \\ b - c &= x + y + z \\ 0 &= 6x + 5y + 4z \end{aligned}$$
is consistent. This is the case if and only if $6x + 5y + 4z = 0$, which is a plane R^3 and it is the required subspace spanned by the given set of vectors.

16. Show that the complex numbers $2 + 3i$ and $1 - 2i$ generate the vector space C over R.

Sol. Let $x + iy = a(2 + 3i) + b(1 - 2i), \quad a, b \in R$
$= (2a + b) + i(3a - 2b)$

Therefore, $2a + b = x$
and $3a - 2b = y$

Solving these equations for a and b, we find
$$a = \frac{2x + y}{7}, \quad b = \frac{3x - 2y}{7}$$

Thus an arbitrary complex number $x + iy$ is expressible as a linear a combination of the given complex numbers and so $2 + 3i$ and $1 - 2i$ generate C over R.

17. Let S and T be subsets of a vector space V. Show that

(i) $\langle S \rangle \cup \langle T \rangle \subset \langle S \cup T \rangle$

(ii) $\langle S \cap T \rangle \subset \langle S \rangle \cap \langle T \rangle$

Give an example to show that set equality need not hold in either case

Sol.(i) Since $S \subset S \cup T$, each linear combination of vectors from S is also a linear combination of vectors from $S \cup T$ and therefore, by definition $\langle S \rangle \subset \langle S \cup T \rangle$. Similarly,
$$\langle T \rangle \subset \langle S \cup T \rangle$$
Hence $\langle S \rangle \cup \langle T \rangle \subset \langle S \cup T \rangle$

Example: In R^2 let $S = \{(1, 0)\}$, $T = \{(0, 1)\}$
Then $\langle S \rangle = \{k(1, 0) : k \in R\}$
$= x$ – axis
$\langle T \rangle = \{l(0, 1) : l \in R\}$
$= y$ – axis
$S \cup T = \{(1, 0), (0, 1)\}$

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Now any vector $(x, y) \in R^2$ is a linear combination of (1, 0) and (0, 1) because

$$(x, y) = x(1, 0) + y(0, 1).$$

Thus $\langle S \cup T \rangle = R^2$.

But $\langle S \rangle \cup \langle T \rangle = \{k(1, 0)\} \cup \{l(0, 1)\}$, for $k, l \in R$

$$\neq R^2 = \langle S \cup T \rangle.$$

17.(ii) $\langle S \cap T \rangle \subset \langle S \rangle \cap \langle T \rangle$

Sol. $S \cap T \subset S$ and $S \cap T \subset T$

As in (i) above

$$\langle S \cap T \rangle \subset \langle S \rangle$$

and $$\langle S \cap T \rangle \subset \langle T \rangle$$

Therefore, $\langle S \cap T \rangle \subset \langle S \rangle \cap \langle T \rangle$.

Example: In R^2, let $S = \{(0, 0), (0, 1)\}$ and $T = \{(0, 0), (0, 3)\}$

Then $S \cap T = \{(0, 0)\}$

Therefore, $\langle S \cap T \rangle = \{k(0, 0) : k \in R\}$

$$= \{(0, 0)\} \quad (1)$$

$$\langle S \rangle = \{\alpha(0, 0) + \beta(0, 1) : \alpha, \beta \in R\}$$

$$= \{(0, \beta) : b \in R\}$$

Similarly, $\langle T \rangle = \{(0, l) : l \in R\}$

$$\langle S \rangle \cap \langle S \rangle = \{(0, y) : y \in R\} \quad (2)$$

From (1) and (ii), $\langle S \cap T \rangle \neq \langle S \rangle \cap \langle T \rangle$.

EXERCISE 6.2 (Page 251)

1. Determine whether the following vectors in R^4 are linearly independent or linearly dependent.

(i) (1, 3, −1, 4), (3, 8, −5, 7), (2, 9, 4, 23)

Sol. Let $a, b, c \in R$ and suppose that

$$a(1, 3, -1, 4) + b(3, 8, -5, 7) + c(2, 9, 4, 23) = (0, 0, 0, 0) \quad (A)$$

i.e. $(a + 3b + 2c, 3a + 8b + 9c, -a - 5b + 4c, 4a + 7b + 23c) = 0$

Therefore, the system of equations

$$a + 3b + 2c = 0 \quad (1)$$

$$3a + 8b + 9c = 0 \quad (2)$$

$$-a - 5b + 4c = 0 \quad (3)$$

$$4a + 7b + 23c = 0 \quad (4)$$

must have a nontrivial solution for the given vectors to be linearly dependent.

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From (1) and (2), we have

$$\frac{a}{27-16} = \frac{b}{6-9} = \frac{c}{8-9} = k \text{ (say)}$$

Then $a = 11k,\ b = -3k,\ c = -k$

Substituting these values of a, b, c into (3) and (4), we find

$$-11k + 15k - 4k = 0$$
$$44k - 21k - 23k = 0$$

Thus (3) and (4) are also satisfied for the values of a, b, c obtained from (1) and (2). Hence the equation (A) holds for scalars a, b, c not all zero and so the given vectors are linearly dependent.

1.(ii) $(1, -2, 4, 1), (2, 1, 0, -3), (1, -6, 1, 4)$.

Sol. Let (for $a, b, c \in R$)

$$a(1, -2, 4, 1) + b(2, 1, 0, -3) + c(1, -6, 1, 4) = 0$$

i.e., $(a + 2b + c, -2a + b - 6c, 4a + c, a - 3b + 4c) = 0$

or
$$a + 2b + c = 0 \quad (1)$$
$$-2a + b - 6c = 0 \quad (2)$$
$$4a + c = 0 \quad (3)$$
$$a - 3b + 4c = 0 \quad (4)$$

From (1) and (2), we have

$$\frac{a}{-13} = \frac{b}{4} = \frac{c}{5} = k \text{ (say)}$$

or $a = -13k,\ b = 4k,\ c = 5k$

Substituting there values of a, b, c into (3) and (4) we find that they are satisfied only when $k = 0$. Thus, the only solution of equations (1) – (4) is $a = b = c = 0$. Hence the given vectors are linearly independent.

2. Let $V = P_3(x)$ be the vector space of all polynomials of degree ≤ 3 over R together with the zero polynomial. Determine whether $u, v, w \in V$ are linearly dependent or linearly independent.

(i) $u = x^3 - 4x^2 + 2x + 3,\ v = x^3 + 2x^2 + 4x - 1,\ w = 2x^3 - x^2 + 3x + 3$

Sol. Let $a, b, c \in R$ and

$$au + bv + cw = 0, \text{ (0 on the right is the zero polynomial)}$$

i.e., $a(x^3 - 4x^2 + 2x + 3) + b(x^3 + 2x^2 + 4x - 1) + c(2x^3 - x^2 - 3x + 3) = 0$

$$(a+b+2c)x^3 + (-4a+2b-c)x^2 + (2a+4b-3c)x + (3a-b+3c) = 0$$

Therefore,

$$a + b + 2c = 0 \quad (1)$$
$$-4a + 2b - c = 0 \quad (2)$$
$$2a + 4b - 3c = 0 \quad (3)$$
$$3a - b + 3c = 0 \quad (4)$$

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From (1) and (2), we have

$$\frac{a}{5} = \frac{b}{-7} = \frac{c}{6} = k \text{ (say)}$$

or $a = -5k, \quad b = -7k, \quad c = 6k$

Substituting these values of a, b, c into (3) and (4) we find that these equations are satisfied only if $k = 0$. Thus $a = b = c = 0$ and so the given vectors are linearly independent.

2 (ii) $u = x^3 - 3x^2 - 2x + 3, v = x^3 - 4x^2 - 3x + 4, w = 2x^3 - 7x^2 - 7x + 9$

Sol. Let $a, b, c \in R$ and $au + bv + cw = 0$

or $a(x^3 - 3x^2 - 2x + 3) + b(x^3 - 4x^2 - 3x + 4) + c(2x^3 - 7x^2 - 7x + 9) = 0$

$$(a + b + 2c)x^3 + (-3a - 4b - 7c)x^2 + (-2a - 3b - 7c)x + (3a + 4b + 9c) = 0$$

Hence
$$a + b + 2c = 0 \quad (1)$$
$$3a + 4b + 7c = 0 \quad (2)$$
$$2a + 3b + 7c = 0 \quad (3)$$
$$3a + 4b + 9c = 0 \quad (4)$$

From (1) and (2), we have

$$\frac{a}{-1} = \frac{b}{-1} = \frac{c}{1} = k \text{ (say)}$$

or $a = k, \quad b = k, \quad c = -k$

Substituting these values of a, b, c into (3) and (4) we find that thee two equations are satisfied only when $k = 0$. Thus the only solution of the system of equations (1) – (4) is $a = b = c = 0$. Hence the given vectors are linearly independent.

3. Show that the vectors $(1 - i, i)$ and $(2, -1 + i)$ in C^2 are linearly dependent over C but linear independent over R.

Sol. Let $a, b \in C$ and

$$a(1 - i, i) + b(2, -1 + i) = 0$$

or $a(1 - i) + 2b = 0 \quad (1)$

and $ai + b(-1 + i) = 0 \quad (2)$

Set $b = 1$ into (1) so that $a = \frac{-2}{1 - i}$

Substituting these values into the L.H.S. of (2), we have

$$\frac{-2i}{1 - i} + (-1 + i) = \frac{-2i - (1 - i)^2}{1 - i} = \frac{-2i - (1 - 1 - 2i)}{1 - i} = 0$$

Thus $a = \frac{-2}{1 - i}$, $b = 1$ is also a solution of (2).

Since a and b are nonzero, the given vectors are linearly dependent over C.

Next suppose $p, q \in R$ and $p(-1, i) + q(2, -1 + i) = 0$

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Thus $p(1-i)+2q = 0$

and $pi+q(-1+i) = 0$

i.e., $p+2q-pi = 0$ (1)

and $-q+(p+q)i = 0$ (2)

From (1), $p+2q=0$ and $p=0$

From (2), $q=0$ and $p+q=0$.

Hence $p=0$ and $q=0$

So the given vectors are linearly independent over R.

4. Show that the vectors $(3+\sqrt{2}, 1+\sqrt{2})$ and $(7, 1+2\sqrt{2})$ in R^2 are linearly dependent over R but linearly independent over Q.

Sol. Let $a, b \in R$ and $a(3+\sqrt{2}, 1+\sqrt{2}) + b(7, 1+2\sqrt{2}) = 0$

Then $(3+\sqrt{2})a + 7b = 0$ (1)

and $(1+\sqrt{2})a + (1+2\sqrt{2})b = 0$ (2)

Let $b = 1$, then (1) $\Rightarrow (3+\sqrt{2})a + 7 = 0$ or $a = -\dfrac{7}{3+\sqrt{2}}$

Substituting $a = -\dfrac{7}{3+\sqrt{2}}$ and $b = 1$ into the L.H.S. of (2) we find

$$(1+\sqrt{2})\left(-\frac{7}{3+\sqrt{2}}\right) + (1+2\sqrt{2})(1)$$

$$= \frac{-7-7\sqrt{2}}{3+\sqrt{2}} + \frac{1+2\sqrt{2}}{1}$$

$$= \frac{-7-7\sqrt{2}+}{3+} \quad \frac{6}{} \quad \frac{2+\sqrt{2}+4}{}$$

$$= \frac{-7-7\sqrt{2}+7+7\sqrt{2}}{3+\sqrt{2}} = \frac{0}{3+\sqrt{2}} = 0 = \text{R.H.S.}$$

Thus $a = -\dfrac{7}{3+\sqrt{2}}$ and $b = 1$ is a solution of both the equations (1) and (2).

Hence we can find nonzero $a, b \in R$ such that

$$a(3+\sqrt{2}, 1+\sqrt{2}) + b(7, 1+2\sqrt{2}) = 0$$

and so the given vectors are linearly dependent over R.

Now suppose $c, d \in Q$ and $c(3+\sqrt{2}, 1+\sqrt{2}) + d(7, 1+2\sqrt{2}) = (0, 0)$

Then $(3+\sqrt{2})c + 7d = 0$

and $(1+\sqrt{2})c + (1+2\sqrt{2})d = 0$

i.e. $(3c+7d) + \sqrt{2}c = 0$ (3)

and $(c+d) + (c+2d)\sqrt{2} = 0$ (4)

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Clearly L.H.S. of each of (3) and (4) is a surd. This means $3c + 7d = 0$, $c = 0$ and $c + d = 0$, $c + 2d = 0$. Thus $c = 0$, $d = 0$ Hence the given vectors.

$$(3+\sqrt{2}, 1+\sqrt{2}), (7, 1+2\sqrt{2})$$

are linearly independent over Q.

5. Suppose u, v and w are linearly independent vectors. Prove that

(i) $u + v - 2w$, $u - v - w$, $u + w$ are linearly independent.

Sol. Suppose a, b, c are scalars and

$$a(u + v - 2w) + b(u - v - w) + c(u + w) = 0$$

or $(a + b + c)u + (a - b)v + (-2a - b + c)w = 0$

Since u, v and w are linearly independent, we have

$$a + b + c = 0 \quad (1)$$
$$a - b = 0 \quad (2)$$
$$-2a - b + c = 0 \quad (3)$$

From (2), we find $a = b$.

Then the remaining two equations reduce to

$$2b + c = 0$$
$$-3b + c = 0$$

Subtracting, we get $5b = 0$, i.e., $b = 0$

Then $a = 0$ and $c = 0$

Hence the given vectors are linearly independent.

5.(ii) $u + v - 3w$, $u + 3v - w$, $u + w$ are linearly independent

Sol. Suppose a, b, c are scalars and $a(u+v-3w)+b(u+3v-w)+c(u+w) = 0$

or $(a + b + c)u + (a + 3b)v + (-3a - b + c)w = 0$

Since u, v and w are linearly independent, we have

$$a + b + c = 0 \quad (1)$$
$$a + 3b = 0 \quad (2)$$
$$-3a - b + c = 0 \quad (3)$$

Subtracting (3) from (1), we have

$$4a + 2b = 0$$

i.e. $2a + b = 0 \quad (4)$

Solving (2) and (4) we find $a = b = 0$. Hence $c = 0$.

Thus the given vectors are linearly independent.

6. Determine k so that the vectors $(1, -1, k - 1)$, $(2, k, -4)$, $(0, 2, +k, -8)$ in R^3 are linearly dependent.

Sol. The given vector are linearly dependent if one of them is a linear combination of the other two. Suppose

$$(1, -1, k - 1) = a(2, k, -4) + b(0, 2 + k, -8), \text{ for } a, b \in R$$

or $(1, -1, k - 1) = (2a, ka + (2 + k)b, -4a - 8b)$.

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Then $2a = 1$ (1)

$ka + (2 + k)b = -1$ (2)

$-4a - 8b = k - 1$ (3)

From (1), $a = \frac{1}{2}$

From (2), $\frac{k}{2} + (k + 2)b = -1$ (4)

From (3), $-4\left(\frac{1}{2}\right) - 8b = k - 1$

or $b = -\frac{k+1}{8}$

Substituting the value of b into (4), we get $k^2 - k - 6 = 0$

or $(k - 3)(k + 2) = 0$

Hence $k = 3, -2$

7. Using the technique of Theorem 6.21 of casting out vectors which are linear combination of others, find a linearly independent subset of the given set spanning the same subspace

(i) $\{(1, -3, 1), (2, 1, -4), (-2, 6, -2), (-1, 10, -7)\}$ in R^3.

Sol. We note that $(-2, 6, -2) = -2(1, -3, 1)$, so we cast out $(-2, 6, -2)$ and obtain the subset $\{(1, -3, 1), (2, 1, -4), (-1, 10, -7)\}$.

Suppose $(-1, 10, -7) = a(1, -3, 1) + b(2, 1, -4)$, where $a, b \in R$. Then

$a + 2b = -1$ (1)

$-3a + b = 10$ (2)

$a - 4b = -7$ (3)

and we check whether this system of equations has a simultaneous solution.

From (1) and (2), we have

$$\frac{a}{-21} = \frac{b}{7} = \frac{1}{7}$$

Thus $a = -3$, $b = 1$. These values satisfy (3). Hence $(-1, 10, -7)$ is a linear combination of the two vectors $(1, -3, 1)$ and $(2, 1, -4)$. We cast out $(-1, 10, -7)$ and are left with the subset

$$S = \{(1, -3, 1), (2, 1, -4)\}$$

Since $(2, 1, -4)$ is not a multiple of $(1, -3, 1)$, the set S is linearly independent which spans the same subspace as the given set of four vectors.

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7.(ii) $\{1, \sin^2 x, \cos 2x, \cos^2 x\}$ in the space of all functions from R to R.

Sol. Since $\cos 2x = \cos^2 x - \sin^2 x$, we cast out $\cos 2x$ and obtain the subset

$\{1, \sin^2 x, \cos^2 x\}$

Again, $\cos^2 x = 1 - \sin^2 x$ so that $\cos^2 x$ is a linear combination of $1, \sin^2 x$. We cast out $\cos^2 x$ and are left with the subset.

$$\{1, \sin^2 x\}$$

This is clearly a linear independent set.

7.(iii) $\{1, 3x - 4, 4x + 3, x^2 + 2, x - x^2\}$ in the space $P_2(x)$ of polynomials.

Sol. Since $3x - 4 = \frac{3}{4}(4x + 3) - \frac{25}{4}(1)$,

we cast out $3x - 4$ and obtain the subset

$$\{1, 4x + 3, x^2 + 2, x - x^2\}.$$

Now $$x - x^2 = -(x^2 + 2) + \frac{1}{4}(4x + 3) + \frac{5}{4}(1),$$

so we cast out $x - x^2$ and are left with the subset

$$S = \{1, 4x + 3, x^2 + 2\}.$$

We check whether S is linearly independent. Let $a, b, c \in R$ and

$$a(1) + b(4x + 3) + c(x^2 + 2) = 0$$

i.e., $$(a + 3b + 2c) + 4bx + cx^2 = 0$$

This implies $$a + 3b + 2c = 0$$
$$4b = 0$$
$$c = 0$$

The only solution of these equations is $a = b = c = 0$ and hence the set S is linearly independent which spans the same subspace as the given set of five vectors.

8. Verify that the polynomials $2 - x^2$, $x^3 - x$, $2 - 3x^2$ and $3 - x^3$ form a basis for $P_3(x)$. Express each of (i) $1 + x$ and (ii) $x + x^2$ as a linear combination of these basis vectors.

Sol.(i) Consider

$$a(2 - x^2) + b(x^3 - x) + c(2 - 3x^2) + d(3 - x^3) = 0$$

i.e., $$(2a + 2c + 3d) - bx + (-a - 3c)x^2 + (b - d)x^3 = 0$$

Then
$$2a + 2c + 3d = 0 \quad (1)$$
$$b = 0 \quad (2)$$
$$-a - 3c = 0 \quad (3)$$
and
$$b - d = 0 \quad (4)$$

From (2) and (4), we have $b = 0 = d$.

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From (1) and (3), we get

$$2a + 2c = 0$$

and $$-a - 3c = 0$$

giving $a = 0 = c$.

Thus the given set of polynomials is linearly independent. Since $P_3(x)$ is of dimension 4, the given vectors form a basis for $P_3(x)$.

Let $1 + x = a(2 - x^2) + b(x^3 - x) + c(2 - 3x^2) + d(3 - x^3)$

$= (2a + 2c + 3d) - bx + (-a - 3c)x^2 + (b - d)x^3$,

where $a, b, c, d \in R$

As before, equating coefficients we have

$$2a + 2c + 3d = 1$$
$$-b = 1$$
$$-a - 3c = 0$$
$$b - d = 0.$$

Clearly $b = d = -1$

$$2a + 2c - 3 = 1 \qquad (1)$$
$$-a - 3c = 0 \quad \text{or} \quad a = -3c$$

Putting this values of a into (1), we get

$$-6c + 2c = 4 \quad \text{or} \quad c = -1, \quad a = -3c = 3$$

Thus $1 + x = 3(2 - x^2) - (x^3 - x) - (2 - 3x^2) - (3 - x^3)$

8.(ii) Let $x + x^2 = a(2 - x^2) + b(x^3 - x) + c(2 - 3x^2) + d(3 - x^3)$

$= (2a + 2c + 3d) - bx + (-a - 3c)x^2 + (b - d)x^3$,

where $a, b, c, d \in R$

Equating coefficients of like terms, we obtain

$$2a + 2c + 3d = 0$$
$$-b = 1$$
$$-a - 3c = 1$$
$$b - d = 0.$$

Solving these equations, we find $b = d = -1$.

$$2a + 2c - 3 = 0$$

and $$-a - 3c - 1 = 0$$

These two equations yield

$$c = \frac{-5}{4} \quad \text{and} \quad a = \frac{11}{4}$$

Therefore, $x + x^2 = \frac{11}{4}(2 - x^2) - (x^3 - x) - \frac{5}{4}(2 - 3x^2) - (3 - x^3)$.

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9. Determine whether or not the given set of vectors is a basis for R^2:

(i) $\{(1, 1), (3, 1)\}$

Sol. Let $a, b \in R$ and $a(1, 1) + b(3, 1) = 0$

i.e., $(a + 3b,\ a + b) = (0, 0)$

Then $a + 3b = 0$

$a + b = 0$

The only simultaneous solution of these equations is $a = b = 0$. Thus the given vectors are linearly independent and they form a basis for R^2.

9.(ii) $\{(2, 1), (1, -1)\}$

Sol. Since the vectors (2, 1) is not a multiple of (1, −1, the given vectors are linearly independent. They also form a basis for R^2

10. Determine whether or not the given set of vectors is a basis for R^3.

(i) $\{(1, 2, -1), (0, 3, 1), (1, -5, 3)\}$

Sol. Let $a, b, c \in R$ and

$$a(1, 2, -1) + b(0, 3, 1) + c(1, -5, 3) = 0$$

i.e., $(a + c,\ 2a + 3b - 5c,\ -a + b + 3c) = 0$

Therefore,
$$a + c = 0 \quad (1)$$
$$2a + 3b - 5c = 0 \quad (2)$$
$$-a + b + 3c = 0 \quad (3)$$

From (2) and (3), we get

$$\frac{a}{14} = \frac{b}{-1} = \frac{c}{5} = k \text{ (say)}$$

i.e., $a = 14k$, $c = 5k$. These values satisfy (1) only if $k = 0$.

Thus a simultaneous solution of equations (1) – (3) is $a = b = c = 0$. Therefore, the given vectors are linearly independent and they form a basis for R^3 since dim $R^3 = 3$

10.(ii) $\{(2, 4, -3), (0, 1, 1), (0, 1, -1)\}$

Sol. Suppose $a, b, c \in R$

and $a(2, 4, -3) + b(0, 1, 1) + c(0, 1, -1) = 0$

or $(2a, 4a, b - c, -3a + b - c) = (0, 0, 0)$

Thus, $2a = 0 \Rightarrow a = 0$

$4a + b + c = 0 \Rightarrow b + c = 0$

$-3a + b - c = 0 \Rightarrow b - c = 0$

Thus the three equations are satisfied only when $a = b = c = 0$

Therefore the given set is linearly independent and it is a basis of R^3

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11. Let V be the real vector space of all functions defined on R into R. Determine whether the given vectors are linearly independent or linearly dependent in V.

(i) $x, \cos x$

Sol. Let $a, b \in R$ and $ax + b\cos x = 0$ for all $x \in R$

Putting $x = 0, \frac{\pi}{2}$ successively, we have

$$b = 0$$

and $$a\frac{\pi}{2} = 0$$

Thus $a = 0$, $b = 0$ and so the vectors are linearly independent.

11.(ii) $\sin^2 x, \cos^2 x, \cos 2x$

Sol. **Method I.** Since $\cos 2x = \cos^2 x - \sin^2 x$ is a linear combination of the other two vectors, the given vectors are linearly dependent.

Method II: For $a, b, c, \in R$, let

$$a\sin^2 x + b\cos^2 x + c\cos 2x = 0 \text{ for all } x \in R$$

Set $x = 0, \frac{\pi}{4}, \frac{\pi}{2}$ in the above equation to get the system of equations:

$$b + c = 0$$

$$\frac{a}{2} + \frac{b}{2} - c = 0$$

$$a - c = 0$$

This system has non-trivial solution

$$(c, -c, c)$$

and so the vectors are linearly dependent.

11.(iii) $\sin x, \cos x, \sinh x, \cosh x$

Sol. Let for $a, b, c, d \in R$ and all $x \in R$,

$$a\sin x + b\cos x + c\sinh x + d\cosh x = 0 \quad \text{(A)}$$

We set $x = 0$ in the equation (A), to have

$$b + d = 0 \quad (1)$$

Differentiating (A) w.r.t. x, we get

$$a\cos x - b\sin x + c\cosh x + d\sinh x = 0 \quad \text{(B)}$$

Put $x = 0$ in the above equation to obtain

$$a + c = 0 \quad (2)$$

Differentiating (B) w.r.t. x, we have (C)

$$-a\sin x - b\cos x + c\sinh x + d\cosh x = 0$$

Putting $x = 0$ into (C), we get

$$-b + d = 0 \quad (3)$$

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Again differentiating (C) and setting $x = 0$ in the result, we obtain

$$-a = 0 \qquad (4)$$

From equations (1) – (4), we have

$$a = 0 = b = c = d$$

and so the vectors are linearly independent.

11.(iv) $\sin x,\ \sin x + \cos x,\ \sin x - \cos x$

Let $a \sin x + b(\sin x + \cos x) + c(\sin x - \cos x) = 0$ (1)

for all $x \in R$ and $a, b, c,$ being some real numbers.

Put $x = 0, \frac{\pi}{2}, \pi$ successively into (1) to get

$$\begin{aligned} b - c &= 0 \\ a + b + c &= 0 \\ -b + c &= 0 \end{aligned}$$

This system has non-trivial solution $(-2c, c, c)$ and so the vectors are linearly dependent.

11.(v) e^{ax}, e^{bx}, e^{cx}; a, b, c being distinct real numbers

Sol. Let $\alpha e^{ax} + \beta e^{bx} + \gamma e^{cx} = 0$ (1)

for all $x \in R$ and α, β, γ being some real numbers.

Set $x = 0, 1, -1$ successively into (1). Then

$$\left.\begin{aligned} \alpha + \beta + \gamma &= 0 \\ \alpha e^{a} + \beta e^{b} + \gamma e^{c} &= 0 \\ \alpha e^{-a} + \beta e^{-b} + \gamma e^{-c} &= 0 \end{aligned}\right\} \qquad (A)$$

The matrix of coefficients of the above system of equations in the variables α, β, γ is

$$A = \begin{bmatrix} 1 & 1 & 1 \\ e^{a} & e^{b} & e^{c} \\ e^{-a} & e^{-b} & e^{-c} \end{bmatrix}$$

$$\text{Det } A = e^{b-c} - e^{-(b-c)} + e^{c-a} - e^{-(c-a)} + e^{a-b} - e^{-(a-b)} \neq 0,$$

since a, b, c are distinct real numbers. Thus the system of equations (A) has the trivial solution $\alpha = \beta = \gamma = 0$ and so the given set of vectors is linearly independent.

12. Determine a basis for each of the following subspaces of R^3:

(i) The plane $x - 2y + 5z = 0$

Sol. The general solution of

$$x - 2y + 5z = 0 \quad \text{is}$$

$$x = 2y - 5z, \quad \text{where y and z are free variables.}$$

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This can be written in vector form as

$$(x, y, z) = (2y - 5z, y, z)$$
$$= y(2, 1, 0) + z(-5, 0, 1)$$

Thus the plane in spanned by the vectors (2, 1, 0), (−5, 0, 1) and these form a basis for the plane since its dimension is two (as a subspace of R^3).

12.(ii) The line $\frac{x}{-2} = \frac{y}{1} = \frac{z}{6}$

Sol. Parametric equations of the given line are

$$x = -2t$$
$$y = t$$
$$z = 6t$$

Thus $(x, y, z) = (-2t, t, 6t)$
$= t(-2, 1, 6)$

The given line is spanned by (−2, 1, 6). Since the dimension of a line (as a subspace of R^3) is 1, the vector (−2, 1, 6) forms a basis for the line.

12.(iii) All vectors of the form (a, b, c), where $3a - 2b + c = 0$

Sol. The general solution of $3a - 2b + c = 0$ is

$$c = 2b - 3a$$

where a and b are free variables. In vector form, this can be written as

$$(a, b, c) = (a, b, 2b - 3a)$$
$$= a(1, 0, -3) + b(0, 1, 2).$$

The space is spanned by the vectors (1, 0, −3) and (0, 1, 2).

Now for any real numbers r and s

$$r(1, 0, -3) + s(0, 1, 2) = (0, 0, 0)$$

implies $(r, s, 2s - 3r) = (0, 0, 0)$

i.e. $r = 0$
$s = 0$
and $2s - 3r = 0$

Thus the vectors (1, 0, −3) and (0, 1, 2) are linearly independent and therefore, they form a basis for the given space.

13. Find the dimension of the subspace

$\{(x_1, x_2, x_3, x_4) : x_2 = x_3\}$ of R^4. Also determine a basis.

Sol. A general vector of the given subspace is

$$(x_1, x_2, x_2, x_4)$$

It can be expressed as a linear combination; i.e.,

$$(x_1, x_2, x_2, x_4) = (x_1, 0, 0, 0) + (0, x_2, x_2, 0), (0, 0, 0, x_4)$$
$$= x_1(1, 0, 0, 0) + x_2(0, 1, 1, 0) + x_4(0, 0, 0, 1).$$

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The set $S = \{(1, 0, 0, 0), (0, 1, 1, 0), (0, 0, 0, 1)\}$ generates the given subspace.

We check whether this set is linearly independent.

For $a, b, c \in R$, let

$$a(1, 0, 0, 0) + b(0, 1, 1, 0) + c(0, 0, 0, 1) = 0$$

Then $a = 0,\ b = 0,\ c = 0$. Thus the set S is linearly independent.

Hence the dimension of the given subspace is 3 and the set S is a basis.

14. A subspace U of R^4 is spanned by the vectors $(1, 0, 2, 3)$ and $(0, 1, -1, 2)$ and a subspace W is spanned by $(1, 2, 3, 4)$, $(-1, -1, 5, 0)$ and $(0, 0, 0, 1)$ Find the dimensions of U and W.

Sol. For $a, b \in R$, let

$$a(1, 0, 2, 3) + b(0, 1, -1, 2) = 0$$

i.e., $(a, b, 2a - b, 3a + 2b) = (0, 0, 0, 0)$

Hence $a = 0,\ b = 0$.

Thus the given vectors spanning U are linearly independent. The dimension of U is 2.

Again for $a, b, c \in R$, let

$$a(1, 2, 3, 4) + b(-1, -1, 5, 0) + c(0, 0, 0, 1) = 0$$

i.e., $(a - b, 2a - b, 3a + 5b, 4a + c) = (0, 0, 0, 0)$

Then
$$a - b = 0 \quad (1)$$
$$2a - b = 0 \quad (2)$$
$$3a + 5b = 0 \quad (3)$$
$$4a + c = 0 \quad (4)$$

From (1), we have $a = b$

Substituting $a = b$ into (2), we get $b = 0$. Thus $a = 0$.

From (4), we have $c = 0$.

The given vectors spanning W are linearly independent and dim $W = 3$.

15. Suppose U and W are distinct four dimensional subspaces of a vector space V of dimension six. Find the possible dimension of $U \cap W$.

Sol. We are given dim $V = 6$, dim $U = 4$, dim $W = 4$

Since $U + W$ is a subspace of V, dim $(U + W) \leq$ dim $V = 6$

Since $U \subseteq U + W$ and $W \subseteq U + W$, dim $U \leq$ dim $(U + W) \leq 6$

i.e., $4 \leq$ dim $(U + W) \leq 6$

Thus dim $(U + W)$ is 4 or 5 or 6.

Since U and W are distinct, they must differ by at least one generator, so

dim $(U + W) > 4$.

Thus possible values of dim $(U + W)$ are 5 or 6.

Now dim $(U + W)$ = dim U + dim W − dim $(U \cap W)$

i.e., $\dim(U \cap W) = \dim U + \dim W - \dim(U + W)$

If $\dim(U + W) = 5$, then

$\dim(U \cap W) = 4 + 4 - 5 = 3$

If $\dim(U + W) = 6$, then

$\dim(U \cap W) = 4 + 4 - 6 = 2$.

16. Find a basis and dimension of the subspace W of R^4 spanned by

(i) $(1, 4, -1, 3), (2, 1, -3, -1)$ and $(0, 2, 1, -5)$

Sol. Let $a, b, c \in R$ and

$$a(1, 4, -1, 3) + b(2, 1, -3, -1) + c(0, 2, 1, -5) = 0$$

i.e., $(a + 2b, 4a + b + 2c, -a - 3b + c, 3a - b - 5c) = (0, 0, 0, 0)$

Then

$$a + 2b = 0 \quad (1)$$
$$4a + b + 2c = 0 \quad (2)$$
$$-a - 3b + c = 0 \quad (3)$$
$$3a - b - 5c = 0 \quad (4)$$

From (1) and (2), we have

$$\frac{a}{4} = \frac{b}{-2} = \frac{c}{-7} = k \text{ (say)}$$

$$a = 4k, \quad b = -2k, \quad c = -7k$$

Substituting these values into the L.H.S. of (3) and (4), we get

$$-4k + 6k - 7k \neq 0$$

and $12k + 2k + 35k \neq 0$ unless $k = 0$.

Therefore, the only simultaneous solution of equations (1) – (4) is $a = b = c = 0$.

Thus the given vectors are linearly independent and they form a basis of the subspace whose dimension is 3

16.(ii) $(1, -4, -2, 1), (1, -3, -1, 2)$ and $(3, -8, -2, 7)$

Sol. We first check whether these vectors are linearly independent.

Let $a, b, c \in R$ and

$$a(1, -4, -2, 1) + b(1, -3, -1, 2) + c(3, -8, -2, 7) = 0$$

Then
$$a + b + 3c = 0 \quad (1)$$
$$-4a - 3b - 8c = 0 \quad (2)$$
$$-2a - b - 2c = 0 \quad (3)$$
$$a + 2b + 7c = 0 \quad (4)$$

From (1) and (2), we have

$$\frac{a}{1} = \frac{b}{-4} = \frac{c}{1} = k \text{ (say)}$$

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Therefore,

$$a = k, \quad b = -4k, \quad c = k$$

These values of a, b, c also satisfy both (3) and (4). Thus the system of equations (1) – (4) has a nontrivial solution and so the given vectors are linearly dependent

Let $(3, -8, -2, 7) = l\,(1, -4, -2, 1) + m(1, -3, -1, 2)$, $l, m \in R$.

Then
$$l + m = 3 \quad (5)$$
$$-4l - 3m = -8 \quad (6)$$
$$-2l - m = -2 \quad (7)$$
$$l + 2m = 7 \quad (8)$$

From (5) and (8), we have $m = 4$ and $l = -1$. These values of l, m also satisfy (6) and (7). Thus (3, –8, –2, 7) is a linear combination of (1, –4, –2, 1) and (1, –3, –1, –2). We cast out (3, –8, –2, 7) and obtain the set

$$S = \{(1, -4, -2, 1)\,(1, -3, -1, 2)\}$$

which span the same subspace as the given three vectors. Since either vector in S is not a multiple of the other, the set S is linearly independent. Thus S is a basis of the subspace and dimension of the subspace is two.

17. Let U and W be 2-dimensional subspaces of R^3. Show that

$$U \cap W \neq \{0\}$$

Sol. $\dim U = 2$, $\dim W = 2$

If $U = W$, then $U \cap W \neq \{0\}$ as $\dim U = \dim W = 2$

Now suppose $U \neq W$. Then generating sets of U and W are not the same. Therefore,

$$\dim (U + W) > 2$$

Since U and W are subspaces of R^3. $U + W$ is a subspace of R^3.

Hence $\dim (U + W) \leq 3$

Thus $2 < \dim (U + W) \leq 3$

This implies $\dim (U + W) = 3$

Now $\dim (U + W) = \dim U + \dim W - \dim (U \cap W)$

i.e., $3 = 2 + 2 - \dim (U \cap W)$

Hence $\dim (U \cap W) = 1$. This mean $U \cap W$ contains a nonzero element and so $U \cap W \neq \{0\}$.

18. If two or more rows (columns) of an $n \times n$ matrix A are linearly dependent, then show that $\det A = 0$.

Sol. Let $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$

be an $n \times n$ matrix whose ith and jth rows are linearly dependent.

$$P_i = [a_{i1} \quad a_{i2} \quad \cdots \quad a_{in}]$$

$$P_j = [a_{j1} \quad a_{j2} \quad \cdots \quad a_{jn}]$$

are row vectors and since they are linearly dependent, there are scalars α and β, not both zero, such that

$$\alpha P_i + \beta P_j = 0.$$

Therefore, $$P_i = \frac{-\beta}{\alpha} P_j$$

$$\det A = -\frac{\beta}{\alpha} \det \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{j1} & a_{j2} & \cdots & a_{jn} \\ \vdots & \vdots & \cdots & \vdots \\ a_{j1} & a_{j2} & \cdots & a_{jn} \\ \vdots & \vdots & \cdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

$$= -\frac{\beta}{\alpha}, 0 = 0$$

since two rows of the determinant are identical. The case when more than two rows are linearly dependent can be tackled in a similar manner. The case of dependent columns is similar.

EXERCISE 6.3 (Page 272)

1. Check which of the following define linear transformations from R^3 to R^2?

(i) $T(x_1, x_2, x_3) = (x_1 - x_2, x_1 - x_3)$

Sol. Let $u = (x_1, x_2, x_3)$, $v = (y_1, y_2, y_3)$

Then $T(u) = (x_1 - x_2, x_1 - x_3)$, $T(v) = (y_1 - y_2, y_1 - y_3)$

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Let $a, b \in R$. Then

$au + bv = (ax_1 + by_1, ax_2 + by_2, ax_3 + by_3)$

$T(au + bv) = T(ax_1 + by_1, ax_2 + by_2, ax_3 + by_3)$

$= (ax_1 + by_1 - ax_2 - by_2, ax_1 + by_1 - ax_3 - by_3)$

$= (a(x_1 - x_2) + b(y_1 - y_2), a(x_1 - x_3) + b(y_1 - y_3))$

$= (a(x_1 - x_2), a(x_1 - x_3)) + (b(y_1 - y_2), b(y_1 - y_3))$

$= aT(u) + bT(v)$

Hence T is a linear transformation from R^3 to R^2.

1.(ii) $T(x_1, x_2, x_3) = (|x_1|, x_2 - x_3)$

Sol. $T(x_1, x_2, x_3) = (|x_1|, x_2 - x_3)$, $T(y_1, y_2, y_3) = (|y_1|, y_2 - y_3)$

Now $T[(x_1, x_2, x_3) + (y_1, y_2, y_3)] = T[(x_1 + y_1, x_2 + y_2, x_3 + y_3)]$

$= (|x_1 + y_1|, x_2 + y_2 - x_3 - y_3)$

And $T(x_1, x_2, x_3) + (y_1, y_2, y_3) = (|x_1|, x_2 - x_3) + (|y_1|, y_2 - y_3)$

$= (|x_1| + |y_1|, x_2 - x_3 + y_2 - y_3)$

$= (|x_1| + |y_1|, x_2 + y_2 - x_3 - y_3)$

We know that $|x_1 + y_1| \leq |x_1| + |y_1|$

and equality holds only if $x_1 = y_1$. Thus $|x_1 + y_1| \neq |x_1| + |y_1|$ in general

i.e., $T[(x_1, x_2, x_3) + (y_1, y_2, y_3)] \neq T(x_1, x_2, x_3) + T(y_1, y_2, y_3)]$

Hence T is not a linear transformation from R^3 to R^2.

1.(iii) $T(x_1, x_2, x_3) = (x_1 + 1, x_2 + x_3)$

Sol. Here $T(0, 0, 0) = (1, 0) \neq (0, 0)$

Thus T is not a linear transformation, because a linear transformation maps the zero vector on to the zero vector.

1.(iv) $T(x_1, x_2, x_3) = (0, x_3)$

Sol. Let $u = (x_1, x_2, x_3)$, $v = (y_1, y_2, y_3)$. Then

$u + v = (x_1 + y_1, x_2 + y_2, x_3 + y_3)$

$T(u + v) = (0, x_3 + y_3)$

$T(u) + T(v) = (0, x_3) + (0, y_3)$

$= (0, x_2 + y_3)$

Thus $T(u + v) = T(u) + T(v)$

Let $a \in R$ Then $T(au) = (0, ax_3) = a(0, x_3) = aT(u)$

Thus T is a linear transformation from R^3 onto R^2.

1.(v) $T(x_1, x_2, x_3) = \left(x_1 + \frac{x_2}{x_3}, x_3\right)$

Sol. In this case $T(x_1, x_2, 0)$ is not defined. So T is not a linear transformation.

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1.(vi) $T(x_1, x_2, x_3) = (3x_1 - 2x_2 + x_3, x_3 - 3x_2 - 2x_1)$.

Sol. Let $u = (x_1, x_2, x_3)$, $v = (y_1, y_2, y_3)$ and $a, b \in R$,

Then $au + bv = (ax_1 + by_1, ax_2 + by_2, ax_3 + by_3)$

$$\begin{aligned} T(au + bv) &= (3(ax_1 + by_1) - 2(ax_2 + by_2) + ax_3 + by_3, \\ &\quad ax_3 + by_3 - 3(ax_2 + by_2) - 2(ax_1 + by_1)) \\ &= (3ax_1 - 2ax_2 + ax_3 + 3by_1 - 2by_2 + by_3, \\ &\quad ax_3 - 3ax_2 - 2ax_1 + by_3 - 3by_2 - 2by_1) \\ aT(u) &= a(3x_1 - 2x_2 + x_3, x_3 - 3x_2 - 2x_1) \\ &= (3ax_1 - 2ax_2 + ax_3, ax_3 - 3ax_2 - 2ax_1) \\ bT(v) &= b(3y_1 - 2y_2 + y_3, y_3 - 3y_2 - 2y_1) \\ &= (3by_1 - 2by_2 + by_3, by_3 - 3by_2 - 2by_1) \\ aT(u) + bT(v) &= (3ax_1 - 2ax_2 + ax_3 + 3by_1 - 2by_2 + by_3, \\ &\quad ax_3 - 3ax_2 - 2ax_1 + by_3 - 3by_2 - 2by_1) \end{aligned}$$

Hence $T(au + bv) = aT(u) + bT(v)$ for all $u, v \in R^3$ and $a, b \in R$.

Thus T is a linear transformation from R^3 into R^2.

2. Show that each of the following defines linear transformation from R^3 to R^3?

(i) $T(x_1, x_2, x_3) = (x_1 - x_2, x_2 - x_3, x_1)$

Sol. Let $u = (x_1, x_2, x_3)$, $v = (y_1, y_2, y_3)$ and $a, b \in R$. Then

$$\begin{aligned} au + bv &= (ax_1 + by_1, ax_2 + by_2, ax_3 + by_3) \\ T(au + bv) &= (ax_1 + by_1 - ax_2 - by_2, ax_2 + by_2 - ax_3 - by_3, ax_1 + by_1) \\ &= (ax_1 - ax_2 + by_1 - by_2, ax_2 - ax_3 + by_2 - by_3, ax_1 + by_1) \\ aT(u) + bT(v) &= a(x_1 - x_2, x_2 - x_3, x_1) + b(y_1 - y_2, y_2 - y_3, y_1) \\ &= (ax_1 - ax_2 + by_1 - by_2, ax_2 - ax_3 + by_2 - by_3, ax_1 + by_1) \end{aligned}$$

Hence $T(au + bv) = aT(u) + bT(v)$ and so T a linear transformation from R^3 into R^3.

2.(ii) $T(x_1, x_2, x_3) = (x_1 + x_2, -x_1 - x_2, x_3)$

Sol. Let $u = (x_1, x_2, x_3) = (y_1, y_2, y_3)$ and $a, b \in R$. Then

$$\begin{aligned} au + bv &= (ax_1 + by_1, ax_2 + by_2, ax_3 + by_3) \\ T(au + bv) &= (ax_1 + by_1 + ax_2 + by_2, -ax_1 - by_1 - ax_2 - by_2, ax_3 + by_3) \\ aT(u) + bT(v) &= a(x_1 + x_2, -x_1 - x_2, x_3) + b(y_1 + y_2, -y_1 - y_2, y_3) \\ &= (ax_1 + ax_2, -ax_1 - ax_2, ax_3) + (by_1 + by_2, -by_1 - by_2, by_3) \\ &= (ax_1 + ax_2 + by_1 + by_2, -ax_1 - ax_2 - by_1 - by_2, ax_3 + by_3) \end{aligned}$$

Hence $T(au + bv) = aT(u) + bT(v)$

So T a linear transformation from R^3 into R^3.

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2.(iii) $T(x_1, x_2, x_3) = (x_2, -x_1, -x_3)$

Sol. Let $u = (x_1, x_2, x_3)$, $v = (y_1, y_2, y_3) \in R^3$, $a, b \in R$

$au + bv = (ax_1 + by_1, ax_2 + by_2, ax_3 + by_3)$

$T(au + bv) = (ax_2 + by_2, -ax_1 - by_1, -ax_3 - by_3)$

$aT(u) + b(Tv) = a(x_2, -x_1, -x_3) + b(y_2, -y_1, -y_3)$

$= (ax_2, -ax_1, -ax_3) + (by_2, -by_1, -by_3)$

$= (ax_2 + by_2, -ax_1 - by_1, -ax_3 - by_3)$

Thus for all $u, v \in R^3$, $a, b \in R$, we have

$T(au + bv) = aT(u) + bT(v)$.

Hence T is a linear transformation from R^3 into R^3.

2.(iv) $T(x_1, x_2, x_3) = (x_1 - 3x_2 - 2x_3, x_2 - 4x_3, x_3)$

Sol. Let $u = (y_1, y_2, y_3)$, $v = (z_1, z_2, z_3) \in R^3$ and $a, b \in R$.

Then $au + bv = (ay_1 + bz_1, ay_2 + bz_2, ay_3 + bz_3)$

$T(au + bv) = (ay_1 + bz_1 - 3(ay_2 + bz_2) - 2(ay_3 + bz_3),$
$ay_2 + bz_2 - 4(ay_3 + bz_3), (ay_3 + bz_3)$

$= (ay_1 - 3ay_2 - 2ay_3 + bz_1 - 3bz_2 - 2bz_3,$
$ay_2 - 4ay_3 + bz_2 - 4bz_3, ay_3 + bz_3)$

$aT(u) + bT(v) = a(y_1 - 3y_2 - 2y_3, y_2 - 4y_3, y_3) + b(z_1 - 3z_2 - 2z_3, z_2 - 4z_3, z_3)$

$= (ay_1 - 3ay_2 - 2ay_3 + bz_1 - 3bz_2 - 2bz_3, ay_2 - 4ay_3 + bz_2 - 4bz_3$
$, ay_3 + bz_3)$

Hence for all $u, v \in R^3$ and $a, b \in R$, we have

$T(au + bv) = aT(u) + bT(v)$

So T is a linear transformation from R^3 into R^3.

(v) $T(x_1, x_2, x_3) = (x_1 + x_3, x_1 - x_3, x_2)$.

Sol. Let $u = (y_1, y_2, y_3)$, $v = (z_1, z_2, z_3) \in R^3$ and $a, b \in R$

Then $au + bv = (ay_1 + bz_1, ay_2 + bz_2, ay_3 + bz_3)$

$T(au + bv) = (ay_1 + bz_1 + ay_3 + bz_3, ay_1 + bz_1 - ay_3 - bz_3, ay_2 + bz_2)$

$aT(u) + bT(v) = a(y_1 + y_3, y_1 - y_3, y_2) + b(z_1 + z_3, z_1 - z_1, z_2)$

$= (ay_1 + bz_1 + ay_3 + bz_3, ay_1 + bz_1 - ay_3 - bz_3, ay_2 + bz_2)$

Thus for all $u, v \in R^3$, $a, b \in R$, we have

$T(au + bnv) = aT(u) + bT(v)$

So T is a linear transformation from R^3 into R^3.

3. Show that each of the following transformations is not linear:

(i) $T: R^2 \to R$ defined by $T(x_1, x_2) = x_1x_2$

Sol. Suppose $u = (1, 1)$, $v = (1, 1)$

Then $u + v = (2, 2)$ and $T(u + v) = T(2, 2) = 2.2 = 4$

$T(u) + T(v) = 1 + 1 = 2$

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Thus $T(u+v) \neq T(u) + T(v)$

So T is not linear from $R^2 \to R$.

3.(ii) $T: R^2 \to R^3$ defined by $T(x_1, x_2) = (x_1 + 1, 2x_2, x_1 + x_2)$

Sol. $T(0, 0) = (1, 0, 0) \neq (0, 0, 0)$

Thus T is not linear from $R^2 \to R^3$.

3.(iii) $T: R^3 \to R^2$ defined by $T(x_1, x_2, x_3) = (|x_1|, 0)$

Sol: Let $u = (1, 1, 1)$, $v = (-1, 1, 1)$

$$u + v = (0, 2, 2)$$
$$T(u+v) = T(0, 2, 2) = (0, 0)$$
$$T(u) + T(v) = T(1, 1, 1) + T(-1, 1, 1) = (1, 0) + (1, 0) = (2, 0)$$

Thus $T(u+v) \neq T(u) + T(v)$

Hence T is not linear from $R^3 \to R^2$.

3.(iv) $T: R^2 \to R^2$ defined by $T(x_1, x_2) = (x_1^2, x_2^2)$

Let $u = (1, 1)$, $v = (-1, 1)$

$$u + v = (0, 2)$$
$$T(u+v) = T(0, 2) = (0, 4)$$
$$T(u) = (1, 1), \quad T(v) = (1, 1)$$
$$T(u) + T(v) = (1, 1) + (1, 1) = (2, 2)$$

Thus $T(u+v) \neq T(u) + T(v)$

Hence T is not linear from $R^2 \to R^2$.

3.(v) $T: R^3 \to R^3$ defined by $T(x_1, x_2, x_3) = (x_1, x_2, x_3) + (1, 1, 1)$.

Sol. Here $T(0, 0, 0) = (1, 1, 1) \neq (0, 0, 0)$

Hence T is not linear from $R^3 \to R^3$.

4. Determine which of the following transformations are linear:

(a) $T: M_{22} \to R$ defined by

(i) $T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = a + d$ **(ii)** $T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = ad - bc.$

Sol.(i) Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $B = \begin{bmatrix} p & q \\ r & s \end{bmatrix}$

$$T(A+B) = T\left(\begin{bmatrix} a+p & b+q \\ c+r & d+s \end{bmatrix}\right)$$
$$= a + p + d + s = (a + d) + (p + s) = T(A) + T(B)$$

If $k \in R$, then $kA = \begin{bmatrix} ka & kb \\ kc & kd \end{bmatrix}$

$$T(kA) = ka + kd = k(a + d) = kT(A)$$

Hence T is linear.

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4.(a)(ii) $T(A+B) = (a+p)(d+s) - (b+q)(c+r)$

$= ad + as + pd + ps - bc - br - qc - qr$

$= (ad - bc) + (ps - qr) + as + pd - br - qc$

$T(A) + T(B) = ad - bc + ps - qr$

So $T(A+B) \neq T(A) + T(B)$

Thus T is not linear.

(b) $T : P_2(x) \to P_2(x)$ defined by

(i) $T(a + bx + cx^2) = a + (b + c)x + (2a - 3b)x^2$

(ii) $T(a + bx + cx^2) = (a + 1) + bx + cx^2$.

Sol.(i) Let $u = a + bx + cx^2$ and $v = p + qx + rx^2$ be two vectors in $P_2(x)$.

$T(u+v) = T((a+p) + (b+q)x + (c+r)x^2)$

$= a + p + (b + q + c + r)x + (2(a+p) - 3(b+q))x^2$

$= a + p + (b + c)x + (q + r)x + (2a - 3b)x^2 + (2p - 3q)x^2$

$= a + (b + c)x + (2a - 3b)x^2 + p + (q + r)x + (2p - 3q)x^2$

$= T(u) + T(v)$.

If $k \in R$, then

$T(ku) = T(ka + kbx + kcx^2)$

$= ka + (kb + kc) + (2ka - 3kb)x^2$

$= k[a + (b + c)x + (2a - 3b)x^2] = kT(u)$

Thus T is linear.

4(b)(ii) $T(a + bx + cx^2) = (a + 1) + bx + cx^2$.

Sol. Let $u = a + bx + cx^2$ and $v = p + qx + rx^2$ be two vectors in R^2.

$T(u+v) = T((a+p) + (b+q)x + (c+r)x^2)$

$= (a + p + 1) + (b + q)x + (c + r)x^2$

$= (a + 1) + bx + cx^2 + p + qx + rx^2$

$T(u) + T(v) = (a + 1) + bx + cx^2 + (p + 1) + qx + rx^2$

$T(u+v) \neq T(u) + T(v)$

Thus T is not linear.

5. If A is an $m \times n$ matrix, show that $T(x) = Ax$ is a linear transformati

from R^n into R^m, where $x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \in R^n$

Sol. Let $A = [a_{ij}]_{mn}$ and $x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$, $y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$ be any two ve

in R^n. For any nonzero scalars α and β, we have

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$$T(\alpha x + \beta y) = [a_{ij}] \begin{bmatrix} ax_1 + \beta y_1 \\ ax_2 + \beta y_2 \\ \vdots \\ ax_n + bx_n \end{bmatrix}$$

$$= [c_{ij}]_{m \times n}, \text{ where}$$

$$c_{ij} = \sum_{j=1}^{n} a_{ij}(\alpha x_j + \beta y_j)$$

$$= \Sigma(\alpha a_{ij}x_j + \beta a_{ij}y_j) = \alpha\Sigma a_{ij}x_j + \beta\Sigma a_{ij}y_j)$$

Therefore, $[c_{ij}]_{m \times n} = T(\alpha x + \beta y)$

$$= \alpha[a_{ij}]x_i + \beta[a_{ij}]y = \alpha Ax + \beta Ay$$

$$= ATx + \beta Ty$$

Hence T is linear.

6. Determine which of the following linear transformations are one-to-one:

(i) $T: R^2 \to R^3$ defined by $T(x_1, x_2) = (x_1 + x_2, x_1 - x_2, x_1 + 2x_2)$

Sol. Let $u = (x_1, x_2)$, $v = (y_1, y_2)$ be two vector in R^2. Then

$T(u) = (x_1 + x_2, x_1 - x_2, x_1 + 2x_2)$

$T(v) = (y_1 + y_2, y_1 - y_2, y_1 + 2y_2)$

$T(u) = T(v)$ implies

$x_1 + x_2 = y_1 + y_2$ (1)

$x_1 - x_2 = y_1 - y_2$ (2)

$x_1 + 2x_2 = y_1 + 2y_2$ (3)

From (1) and (2) we have

$x_1 = y_1$

Putting this value of y_1 into (3), we find $x_2 = y_2$. Thus $u = v$ and so T is one-to-one.

6.(ii) $T: R^3 \to R^2$ defined by $T(x_1, x_2, x_3) = (x_1 - x_2, x_3)$

Sol. Let $u = (x_1, x_2, x_3)$ and $v = (y_1, y_2, y_3)$ be any two vectors in R^3.

$T(u) = T(v)$ implies

$(x_1 - x_2, x_3) = (y_1 - y_2, y_3)$

i.e. $x_1 - x_2 = y_1 - y_2$ (1)

and $x_3 = y_3$

From (1) we cannot infer that $x_1 = y_1$ and $x_2 = y_2$. Thus $T(u) = T(v)$ does not imply $u = v$ and therefore, T is not one-to-one.

6.(iii) $T: R^2 \to R^3$ defined by $T(x_1, x_2) = (x_1, x_1 + x_2, x_1 - x_2)$.

Sol. Let $u = (x_1, x_2)$ and $v = (y_1, y_2)$ be two vectors in R^2.

$T(u) = T(v)$ implies

$(x_1, x_1 + x_2, x_1 - x_2) = (y_1, y_1 + y_2, y_1 - y_2)$

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or $x_1 = y_1$

$x_1 + x_2 = y_1 + y_2,$

and $x_1 - x_2 = y_1 - y_2$

From the above three equations we have $x_1 = y_1$ and $x_2 = y_2$. Thus $u = v$ and so T is one-to-one.

7. Let C be the vector space of complex numbers over R and $T : C \to C$ be defined by $T(z) = \bar{z}$, where $\bar{z}$ denotes the complex conjugate of z. Show that T is linear.

Sol. Let $z_1, z_2 \in C$ and $a, b \in R$

$$T(az_1 + bz_2) = \overline{az_1 + bz_2}$$

$$= a\bar{z}_1 + b\bar{z}_2, \quad (\text{since } a, b \in R)$$

$$= aT(z_1) + bT(z_2)$$

Hence T is linear.

8. Let V be the vector space $P_n(x)$ of polynomials $p(x)$ with real coefficients and of degree not exceeding n together with the zero polynomial. Let $T : V \to V$ be defined by

$T(p(x)) = p(x + 1)$. Show that T is linear.

Sol. Let $u = p(x)$, $v = q(x)$ and $a, b \in R$

Then $T(au + bv) = T(ap(x) + bq(x))$

$= ap(x + 1) + bq(x + 1)$

$= aT(u) + bT(v)$

Thus T is linear.

9. Let $v_1 = (1, 1, 1)$, $v_2 = (1, 1, 0)$ and $v_3 = (1, 0, 0)$ be a basis for R^3. Find a linear transformation $T : R^3 \to R^2$ such that

$T(v_1) = (1, 0)$, $T(v_2) = (2, -1)$ and $T(v_3) = (4, 3)$

Sol. Let $x = (x_1, x_2, x_3)$ be any vector of R^3. Then

$x = a_1v_1 + a_2v_2 + a_3v_3$, where a_1, a_2, a_3 are real numbers. i.e.

$(x_1, x_2, x_3) = a_1(1, 1, 1) + a_2(1, 1, 0) + a_3(1, 0, 0)$

$= (a_1 + a_2 + a_3, a_1 + a_2, a_1)$

Therefore,

$x_1 = a_1 + a_2 + a_3$

$x_2 = a_1 + a_2$

$x_3 = a_1$

From (2) and (3), we get

$a_2 = x_2 - x_3$

and from (1) and (2), we have

$a_3 = x_1 - x_2$

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Thus $x = (x_1, x_2, x_3)$

$= x_3\nu_1 + (x_2 - x_3)\nu_2 + (x_1 - x_2)\nu_3$

$T(x) = T(x_3\nu_1 + (x_2 - x_3)\nu_2 + (x_1 - x_2)\nu_3)$

$= x_3 T(\nu_1) + (x_2 - x_3) T(\nu_2) + (x_1 - x_2) T(\nu_3)$,

since T is linear

$= x_3 (1, 0) + (x_2 - x_3)(2, -1) + (x_1 - x_2)(4, 3)$

$= (4x_1 - 2x_2 - x_3, 3x_1 - 4x_2 + x_3)$

which is the required linear transformation.

10. Let $T : R^2 \to R$ be the linear transformation for which

$T(1, 1) = 3$ and $T(0, 1) = -2$.

Find $T(x_1, x_2)$.

Sol. The vectors (1, 1), (0, 1) form a basis for R^2. Let (x_1, x_2) be an arbitrary vector of R^2. Then

$(x_1, x_2) = a(1, 1) + b(0, 1)$, where $a, b \in R$

$= (a, a + b)$

Thus $x_1 = a$, $x_2 = a + b$ or $b = x_2 - a = x_2 - x_1$

Therefore, $(x_1, x_2) = x_1 (1, 1) + (x_2 - x_1)(0, 1)$

Applying T on both sides, we have

$T(x_1, x_2) = T(x_1 (1, 1)) + T((x_2 - x_1)(0, 1))$

$= x_1 T(1, 1) + (x_2 - x_1) T(0, 1)$, since T is linear.

$= x_1 (3) + (x_2 - x_1)(-2)$

$= 5x_1 - 2x_2$

expressing T in terms of components.

11. Let $D : P_2(x) \to P_2(x)$ be the differentiation operator

$D(p(x)) = p'(x)$ for all $p(x) \in P_2(x)$. Find $N(D)$.

Sol. $N(D)$ consists of those polynomials in $P_2(x)$ whose derivatives are zero. Since $D(p(x)) = 0$ if $p(x)$ = constant, therefore, $N(D)$ consists of all constant polynomials.

12. Define $T : R^3 \to R^3$ by $T(x_1, x_2, x_3) = (-x_3, x_1, x_1 + x_3)$. Find $N(T)$. Is T one-to-one ?

Sol. $N(T) = \{x = (x_1, x_2, x_3) \in R^3 : T(x_1, x_2, x_3) = (0, 0, 0)\}$

Now $T(x_1, x_2, x_3) = (-x_3, x_1, x_1, x_3) = (0, 0, 0)$

only if $-x_3 = 0$, $x_1 = 0$ and $x_1 + x_3 = 0$

Thus $x_1 = 0$, $x_3 = 0$

$N(T)$ consists of all vectors of the form $(0, x_2, 0)$ which is the x_2-axis.

Since $N(T) \neq (0, 0, 0)$, T is not one-to-one.

13. Suppose U, V and W are vector spaces over the same field F. Let $T : U \to V$ and $S : V \to W$ be linear transformations. The transformation $S \circ T : U \to W$ is defined by $(S \circ T)(u) = S(T(u))$ for all $u \in U$. Show that $S \circ T$ is a linear transformation.

Sol. Let $u_1, u_2 \in U$. Then

$$\begin{aligned}(SoT)(u_1 + u_2) &= S(T(u_1 + u_2)) \\ &= S(T(u_1) + T(u_2)), \text{ since } T \text{ is linear} \\ &= S(T(u_1)) + S(T(u_2)), \text{ since } S \text{ is linear} \\ &= (SoT)(u_1) + (SoT)(u_2)\end{aligned}$$

If $a \in F$, then

$$\begin{aligned}(SoT)(au) &= S(T(au)) \\ &= S(aT(u)), \text{ since } T \text{ is linear} \\ &= aS(T(u)), \text{ since } S \text{ is linear} \\ &= a(SoT)(u)\end{aligned}$$

Thus SoT is a linear transformation.

14. Let U and V be two vector spaces over the same field F. Denote the set of all linear transformations from U into V by $L(U, V)$. Show that $L(U, V)$ is a vector space over F with vector space operations as defined in **Example 33**.

Sol.(i) We first check that $L(U, V)$ is an abelian group under addition.

Let $G, S, T \in L(U, V)$. We prove that + is associative.

$$\begin{aligned}[G + (S + T)](u) &= G(u) + (S + T)(u), u \in U \\ &= G(u) + [S(u) + T(u)] \\ &= [G(u) + S(u)] + T(u) \\ &= (G + S)(u) + T(u) \\ &= [(G + S) + T](u)\end{aligned}$$

Therefore, $G + (S + T) = (G + S) + T$

$$\begin{aligned}(S + T)(u) &= S(u) + T(u) = T(u) + S(u) \\ &= (T + S)(u)\end{aligned}$$

and so $S + T = T + S$.

The zero transformation is the identity of the binary operation +. $(-T)(u) = -T(u)$ and $-T$ is the additive inverse of T. Hence $L(U, V)$ is an abelian group under addition.

Now we check the other axioms of a vector space.

(ii) $[a(S + T)](u) = a(S + T)(u)$, $a \in F$, $u \in U$

$$\begin{aligned}&= a[S(u) + T(u)] = aS(u) + aT(u) \\ &= (aS)(u) + (aT)(u) = (aS + aT)(u)\end{aligned}$$

Therefore, $a(S + T) = aS + aT$

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(iii) $[(a+b)\,T]\,(u) = (a+b)\,T(u), \quad a, b \in F$
$= aT(u) + bT(u) = (aT)(u) + (bT)(u)$
$= (aT + bT)(u)$
Thus, $(a+b)\,T = aT + bT$.

(iv) $[(ab)\,T](u) = (ab)\,T(u) = a(bT(u)) = [a(bT)](u)$
and so $(ab)\,T = a(bT)$

(v) $(1 \cdot T)(u) = 1\,T(u) = T(u)$ showing that
$1 \cdot T = T$.

Thus all axioms of a vector space are satisfied and so $L(U, V)$ is a vector space over F.

15. Find a basis and dimension of each of $R(T)$ and $N(T)$, where

(i) $T : R^3 \to R^3$ is defined by
$T(x_1, x_2, x_3) = (x_1 + 2x_2 - x_3, x_2 + x_3, x_1 + x_2 - 2x_3)$

(ii) $T : R^3 \to R^4$ is defined by
$T(x_1, x_2, x_3) = (2x_1 + x_3, 4x_1 + x_2, x_1 + x_3, x_3 - 4x_2)$.

Sol.(i) R^3 is generated by $(1, 0, 0), (0, 1, 0), (0, 0, 1)$. $R(T)$ will be generated by $T(1, 0, 0), T(0, 1, 0)$ and $T(0, 0, 1)$.

But $T(1, 0, 0) = (1, 0, 1)$
$T(0, 1, 0) = (2, 1, 1)$
$T(0, 0, 1) = (-1, 1, -2)$

Since $(2, 1, 1) = 3(1, 0, 1) + 1(-1, 1, -2)$.

casting out $(2, 1, 1)$, the set $\{(1, 0, 1), (-1, 1, -2)\}$ also spans $R(T)$. This is a linearly independent set and so it is a basis for $R(T)$ and dim $R(T) = 2$.

A vector (x_1, x_2, x_3) is in $N(T)$ if $T(x_1, x_2, x_3) = (0, 0, 0)$

i.e. $(x_1 + 2x_2 - x_3, x_2 + x_3, x_1 + x_2 - 2x_3) = (0, 0, 0)$

or $x_1 + 2x_2 - x_3 = 0$ (1)
$x_2 + x_3 = 0$ (2)
$x_1 + x_2 - 2x_3 = 0$ (3)

From (1) and (2), we have
$x_1 + 3x_2 = 0$
i.e., $x_1 = -3x_2$

From (3), $x_1 + x_2 - 2x_3 = -2x_2 - 2x_3 = 0$ implies
$x_2 = -x_3$

If we take $x_2 = 1$, then $x_1 = -3$ and $x_3 = -1$ so that the vector $(-3, 1, -1)$ spans $N(T)$. Thus dim $N(T) = 1$.

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15.(ii) $R(T)$ is spanned by $T(1, 0, 0)$, $T(0, 1, 0)$, $T(0, 0, 1)$.

Now $T(1, 0, 0) = (2, 4, 1, 0)$

$T(0, 1, 0) = (0, 1, 0, -4)$

$T(0, 0, 1) = (1, 0, 1, 1)$

We check whether these vectors are linearly independent.

Let $a(2, 4, 1, 0) + b(0, 1, 0, -4) + c(1, 0, 1, 1) = 0, a, b, c \in F$

Then
$$2a + c = 0 \quad (1)$$
$$4a + b = 0 \quad (2)$$
$$a + c = 0 \quad (3)$$
$$-4b + c = 0 \quad (4)$$

From (1) and (3), $a = 0$, $c = 0$ and so $b = 0$. Thus the vectors are linearly independent.

A basis for $R(T)$ is

$$\{(2, 4, 1, 0), (0, 1, 0, -4), (1, 0, 1, 1)\}$$

and $\dim R(T) = 3$

Since $\dim R^3 = 3 = \dim N(T) + \dim R(T)$, we have $\dim N(T) = 0$. Thus $N(T) = \{(0, 0, 0)\}$ and so T is a one-to-one linear transformation.

16. Show that linear transformations preserve linear dependence.

Sol. Let $T : U \to V$ be a linear transformation, where U, V are vector spaces over the same field F. Suppose a set

$$\{u_1, u_2, \cdots, u_n\}$$

in U is linearly dependent. We need to show that the set

$$\{T(u_1), T(u_2), \cdots, T(u_n)\}$$

is linearly dependent in V.

Since $\{u_1, u_2, \cdots, u_n\}$ is linearly dependent, there exist scalars $a_1, a_2, \cdots, a_n$, not all zero, such that

$$a_1u_1 + a_2u_2 + \cdots + a_nu_n = 0 \quad (1)$$

Now T is linear and $T(0) = 0$. Applying T to both sides of (1), we have

$$T(a_1u_1 + a_2u_2 + \cdots + a_nu_n) = T(0) = 0$$

or
$$a_1T(u_1) + a_2T(u_2) + \cdots + a_nT(u_n) = 0$$

Since $a_1, a_2, \cdots, a_n$ are not all zero, the above equation shows that the vectors

$$T(u_1), T(u_2), \cdots, T(u_n)$$

are linearly dependent in V. Hence T preserves linear dependence.

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17. Find the rank of each matrix in **Problem 8** of **EXERCISE 3.2** by the method of **(6.42)**.

(i) Let $A = \begin{bmatrix} 1 & 3 \\ 0 & -2 \\ 5 & -1 \\ -2 & 3 \end{bmatrix}$

$$\text{Rank } A = 1 + \text{Rank} \begin{bmatrix} \begin{vmatrix} 1 & 3 \\ 0 & -2 \end{vmatrix} \\ \begin{vmatrix} 1 & 3 \\ 5 & -1 \end{vmatrix} \\ \begin{vmatrix} 1 & 3 \\ -2 & 3 \end{vmatrix} \end{bmatrix}$$

$$A = 1 + \text{Rank} \begin{bmatrix} -2 \\ -16 \\ 9 \end{bmatrix} = 1 + 1 = 2.$$

17.(ii) Let $A = \begin{bmatrix} 1 & 2 & -3 \\ 2 & 1 & 0 \\ -2 & -1 & 3 \\ -1 & 4 & -2 \end{bmatrix}$

$$\text{Rank } A = 1 + \text{Rank} \begin{bmatrix} \begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix} & \begin{vmatrix} 1 & 3 \\ 2 & 0 \end{vmatrix} \\ \begin{vmatrix} 1 & 2 \\ -2 & -1 \end{vmatrix} & \begin{vmatrix} 1 & -3 \\ -2 & 3 \end{vmatrix} \\ \begin{vmatrix} 1 & 2 \\ -1 & 4 \end{vmatrix} & \begin{vmatrix} 1 & -3 \\ -1 & -2 \end{vmatrix} \end{bmatrix}$$

$$= 1 + \text{Rank} \begin{bmatrix} -3 & -6 \\ 3 & -3 \\ 6 & -5 \end{bmatrix}$$

$$= 2 + \text{Rank} \begin{bmatrix} \begin{vmatrix} -3 & -6 \\ 3 & -3 \end{vmatrix} \\ \begin{vmatrix} -3 & -6 \\ 6 & -5 \end{vmatrix} \end{bmatrix} = 2 + \begin{bmatrix} 27 \\ 51 \end{bmatrix} = 2 + 1 = 3,$$

as column rank of $\begin{bmatrix} 27 \\ 51 \end{bmatrix}$ is 1.

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17.(iii) Let $\begin{bmatrix} 1 & 3 & 1 & -2 & -3 \\ 1 & 4 & 3 & -1 & -4 \\ 2 & 3 & -4 & -7 & -3 \\ 3 & 8 & 1 & -7 & -8 \end{bmatrix}$

$$\text{Rank } A = 1 + \text{Rank} \begin{bmatrix} \begin{vmatrix} 1 & 3 \\ 1 & 4 \end{vmatrix} & \begin{vmatrix} 1 & 1 \\ 1 & 3 \end{vmatrix} & \begin{vmatrix} 1 & -2 \\ 1 & -1 \end{vmatrix} & \begin{vmatrix} 1 & -3 \\ 1 & -4 \end{vmatrix} \\ \begin{vmatrix} 1 & 3 \\ 2 & 3 \end{vmatrix} & \begin{vmatrix} 1 & 1 \\ 2 & -4 \end{vmatrix} & \begin{vmatrix} 1 & -2 \\ 2 & -7 \end{vmatrix} & \begin{vmatrix} 1 & -3 \\ 2 & -3 \end{vmatrix} \\ \begin{vmatrix} 1 & 3 \\ 3 & 8 \end{vmatrix} & \begin{vmatrix} 1 & 1 \\ 3 & 1 \end{vmatrix} & \begin{vmatrix} 1 & -2 \\ 3 & -7 \end{vmatrix} & \begin{vmatrix} 1 & -3 \\ 3 & -8 \end{vmatrix} \end{bmatrix}$$

$$= 1 + \text{Rank} \begin{bmatrix} 1 & 2 & 1 & -1 \\ -3 & -6 & -3 & 3 \\ -1 & -2 & -1 & 1 \end{bmatrix}$$

$$= 2 + \text{Rank} \begin{bmatrix} \begin{vmatrix} 1 & 2 \\ -3 & -6 \end{vmatrix} & \begin{vmatrix} 1 & 1 \\ -3 & -3 \end{vmatrix} & \begin{vmatrix} 1 & -1 \\ -3 & 3 \end{vmatrix} \\ \begin{vmatrix} 1 & 2 \\ -1 & -2 \end{vmatrix} & \begin{vmatrix} 1 & 1 \\ -1 & -1 \end{vmatrix} & \begin{vmatrix} 1 & -1 \\ -1 & 1 \end{vmatrix} \end{bmatrix}$$

$$= 2 + \text{Rank} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 2 + 0 = 2.$$

17.(iv) Let $A = \begin{bmatrix} 1 & 3 & -2 & 5 & 4 \\ 1 & 4 & 1 & 3 & 5 \\ 1 & 4 & 2 & 4 & 3 \\ 2 & 7 & -3 & 6 & 13 \end{bmatrix}$

$$\text{Rank } A = 1 + \text{Rank} \begin{bmatrix} \begin{vmatrix} 1 & 3 \\ 1 & 4 \end{vmatrix} & \begin{vmatrix} 1 & -2 \\ 1 & 1 \end{vmatrix} & \begin{vmatrix} 1 & 5 \\ 1 & 3 \end{vmatrix} & \begin{vmatrix} 1 & 4 \\ 1 & 5 \end{vmatrix} \\ \begin{vmatrix} 1 & 3 \\ 1 & 4 \end{vmatrix} & \begin{vmatrix} 1 & -2 \\ 1 & 2 \end{vmatrix} & \begin{vmatrix} 1 & 5 \\ 1 & 4 \end{vmatrix} & \begin{vmatrix} 1 & 4 \\ 1 & 3 \end{vmatrix} \\ \begin{vmatrix} 1 & 3 \\ 2 & 7 \end{vmatrix} & \begin{vmatrix} 1 & -2 \\ 2 & -3 \end{vmatrix} & \begin{vmatrix} 1 & 5 \\ 2 & 6 \end{vmatrix} & \begin{vmatrix} 1 & 4 \\ 2 & 13 \end{vmatrix} \end{bmatrix}$$

$$= 1 + \text{Rank} \begin{bmatrix} 1 & 3 & -2 & 1 \\ 1 & 4 & -1 & -1 \\ 1 & 1 & -4 & 5 \end{bmatrix}$$

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$$= 2 + \text{Rank}\begin{bmatrix} \begin{vmatrix} 1 & 3 \\ 1 & 4 \end{vmatrix} & \begin{vmatrix} 1 & -2 \\ 1 & -1 \end{vmatrix} & \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} \\ \begin{vmatrix} 1 & 3 \\ 1 & 1 \end{vmatrix} & \begin{vmatrix} 1 & -2 \\ 1 & -4 \end{vmatrix} & \begin{vmatrix} 1 & 1 \\ 1 & 5 \end{vmatrix} \end{bmatrix}$$

$$= 2 + \text{Rank}\begin{bmatrix} 1 & 1 & -2 \\ -2 & -2 & 4 \end{bmatrix}$$

$$= 3 + \text{Rank}\begin{bmatrix} \begin{vmatrix} 1 & 1 \\ -2 & -2 \end{vmatrix} & \begin{vmatrix} 1 & -2 \\ -2 & 4 \end{vmatrix} \end{bmatrix}$$

$$= 3 + \text{Rank}\begin{bmatrix} 0 & 0 \end{bmatrix} = 3 + 0 = 3.$$

EXERCISE 6.4 (Page 279)

1. Find the matrix of each of the following linear transformations from R^3 to R^3 with respect to the standard basis for R^3.

(i) $T(x_1, x_2, x_3) = (x_1, x_2, 0)$

Sol. The standard basis of R^3 consists of the vectors

$$e_1 = (1, 0, 0), \quad e_2 = (0, 1, 0), \quad e_3 = (0, 0, 1)$$

Now, $T(e_1) = T(1, 0, 0) = (1, 0, 0) = 1e_1 + 0e_2 + 0e_3$

$T(e_2) = T(0, 1, 0) = (0, 1, 0) = 0e_1 + 1e_2 + 0e_3$

$T(e_3) = T(0, 0, 1) = (0, 0, 0) = 0e_1 + 0e_2 + 0e_3$

Hence the matrix of linear transformation T with respect to the standard basis is

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

1.(ii) $T(x_1, x_2, x_3) = (x_1 + x_2, -x_1 - x_2, x_3)$

Sol. The standard basis of R^3 consists of the vectors

$$e_1 = (1, 0, 0), \quad e_2 = (0, 1, 0), \quad e_3 = (0, 0, 1)$$

Now, $T(e_1) = T(1, 0, 0) = (1, -1, 0) = 1e_1 - 1e_2 + 0e_3$

$T(e_2) = T(0, 1, 0) = (1, -1, 0) = 1e_1 - 1e_2 + 0e_3$

$T(e_3) = T(0, 0, 1) = (0, 0, 1) = 0e_1 + 0e_2 + 1e_3$

Hence the matrix of the linear transformation T with respect to the standard basis is

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

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1.(iii) $T(x_1, x_2, x_3) = (x_2, -x_1, -x_3)$

Sol. Let $T : R^3 \to R^3$. The defining rule of T is given by.

$$T(x_1, x_2, x_3) = (x_2, -x_1, -x_3)$$

The standard basis of R^3 consists of the vectors

$$e_1 = (1, 0, 0), \quad e_2 = (0, 1, 0), \quad e_3 = (0, 0, 1)$$

Now, $T(e_1) = T(1, 0, 0) = (0, -1, 1) = 0e_1 - 1e_2 + 0e_3$

$$T(e_2) = T(0, 1, 0) = (1, 0, 1) = 1e_1 + 0e_2 + 0e_3$$

$$T(e_3) = T(0, 0, 1) = (0, 0, -1) = 0e_1 + 0e_2 - 1e_3$$

Hence the matrix linear transformation T with respect to the standard basis is

$$A = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

1.(iv) $T(x_1, x_2, x_3) = (x_1, x_2 + x_3, x_1 + x_2 + x_3)$.

Sol. The standard basis of R^3 consists of the vectors

$$e_1 = (1, 0, 0), \quad e_2 = (0, 1, 0), \quad e_3 = (0, 0, 1)$$

Now, $T(e_1) = T(1, 0, 0) = (1, 0, 1) = 1e_1 + 0e_2 + 1e_3$

$$T(e_2) = T(0, 1, 0) = (0, 1, 1) = 0e_1 + 1e_2 + 1e_3$$

$$T(e_3) = T(0, 0, 1) = (0, 1, 1) = 0e_1 + 1e_2 + 1e_3$$

Hence the matrix linear transformation T with respect to the standard basis is

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}.$$

2. Find the matrix of each of the following linear transformations with respect to the standard bases of the given spaces:

(i) $T : R \to R^2$ defined by $T(x) = (3x, 5x)$

Sol. The standard basic of R is $\{e = (1)\}$ and the standard basis of R^2 is

$$\{(e_1) = (1, 0), \quad e_2 = (0, 1)\}. \text{ Now}$$

$$T(e) = T(1) = (3, 5) = (3, 0) + (0, 5)$$

$$= 3(1, 0) + 5(0, 1) = 3e_1 + 5e_2$$

Hence the matrix of the linear transformation T with respect to the standard bases is

$$A = \begin{bmatrix} 3 \\ 5 \end{bmatrix}.$$

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2.(ii) $T : R^3 \to R^2$ defined by

$$T(x_1, x_2, x_3) = (3x_1 - 4x_2 + 9x_3, 5x_1 + 3x_2 - 2x_3)$$

Sol. The linear transformation $T : R^3 \to R^2$ is defined by

$$T(x_1, x_2, x_3) = (3x_1 - 4x_2 + 9x_3, 5x_1 + 3x_2 - 2x_3)$$

The standard basis of R^3 is $\{ e_1 = (1, 0, 0),\ e_2 = (0, 1, 0),\ e_3 = (0, 0, 1)\}$ and the standard basis of R^2 is $\{ u_1 = (1, 0),\ u_2 = (0, 1)\}$. Now

$$\begin{aligned} T(e_1) &= T(1, 0, 0) = (3, 5) = (3, 0) + (0, 5) \\ &= 3(1, 0) + 5(0, 1) = 3u_1 + 5u_2 \\ T(e_2) &= T(0, 1, 0) = (-4, 3) = (-4, 0) + (0, 3) \\ &= -4(1, 0) + 3(0, 1) = -4u_1 + 3u_2 \\ T(e_3) &= T(0, 0, 1) = (9, -2) = (9, 0) + (0, -2) \\ &= 9(1, 0) - 2(0, 1) = 9u_1 - 2u_2 \end{aligned}$$

Hence the matrix of linear transformation T with respect to the standard bases is

$$A = \begin{bmatrix} 3 & -4 & 9 \\ 5 & 3 & -2 \end{bmatrix}$$

2.(iii) $T : R^2 \to R^4$ defined by

$$T(x_1, x_2) = (3x_1 + 4x_2, 5x_1 - 2x_2, x_1 + 7x_2, 4x_1)$$

Sol. The standard basis of R^2 is $\{ e_1 = (1, 0),\ e_2 = (0, 1)\}$ and the standard basis of R^4 is $\{ u_1 = (1, 0, 0, 0),\ u_2 = (0, 1, 0, 0),\ u_3 = (0, 0, 1, 0),\ u_4 = (0, 0, 0, 1)\}$

Now $T(e_1) = T(1, 0) = (3, 5, 1, 4) = 3u_1 + 5u_2 + 1u_3 + 4u_2$

$T(e_2) = T(0, 1) = (4, -2, 7, 0) = 4u_1 - 2u_2 + 7u_3 + 0u_2$

Hence the matrix of the linear transformation T with respect to the standard bases is

$$A = \begin{bmatrix} 3 & 4 \\ 5 & -2 \\ 1 & 7 \\ 4 & 0 \end{bmatrix}$$

2.(iv) $T : R^4 \to R$ defined by $T(x_1, x_2, x_3, x_4) = 2x_1 + 3x_2 - 7x_3 + x_4$

Sol. The standard basis of R^4 is

$$\{ e_1 = (1, 0, 0, 0),\ e_2 = (0, 1, 0, 0),\ e_3 = (0, 0, 1, 0),\ e_4 = (0, 0, 0, 1)$$

The standard basis of R is $\{ e = (1)\}$. Now

$$\begin{aligned} T(e_1) &= T(1, 0, 0, 0) = 2 = 2e \\ T(e_2) &= T(0, 1, 0, 0) = 3 = 3e \\ T(e_3) &= T(0, 0, 1, 0) = -7 = -7e \\ T(e_4) &= T(0, 0, 0, 1) = 1 = 1e \end{aligned}$$

Hence the matrix of the linear transformation T with respect to the standard bases is

$$A = [2 \quad 3 \quad -7 \quad 1].$$

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3. Each of the following is the matrix of a linear transformation $T: R^n \to R^m$. Determine m, n and express T is terms of coordinates:

(i) $\begin{bmatrix} 6 & -1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix}$

Sol. Here $A = \begin{bmatrix} 6 & -1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix}$, so $n = 2,\ m = 3$

Let $x = (x_1, x_2)$, then T is given by

$$T(x) = Ax = \begin{bmatrix} 6 & -1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 6x_1 - x_2 \\ x_1 + 2x_2 \\ x_1 + 3x_2 \end{bmatrix}$$

Hence $T(x_1, x_2) = (6x_1 - x_2, x_1 + 2x_2, x_1 + 3x_2)$

3.(ii) $\begin{bmatrix} 1 & 1 & 2 \\ 2 & 5 & 6 \\ -2 & 3 & -1 \end{bmatrix}$

Sol. Here $A = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 5 & 6 \\ -2 & 3 & -1 \end{bmatrix}$ so $n = 3,\ m = 3.$

Let $x = (x_1, x_2, x_3)$, then T is given by

$$T(x) = Ax = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 5 & 6 \\ -2 & 3 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 + x_2 + 2x_3 \\ 2x_1 + 5x_2 + 6x_3 \\ -2x_1 + 3x_2 - x_3 \end{bmatrix}$$

Hence $T(x_1, x_2, x_3) = (x_1 + x_2 + 2x_3, 2x_1 + 5x_2 + 6x_3, -2x_1 + 3x_2 - x_3)$

3.(iii) $\begin{bmatrix} 3 & 1 & 0 & 2 & 0 \\ 1 & 0 & 0 & 1 & 1 \\ 0 & -1 & 1 & 1 & 1 \end{bmatrix}$

Sol. $T: R^n \to R^m$ and $A = \begin{bmatrix} 3 & 1 & 0 & 2 & 0 \\ 1 & 0 & 0 & 1 & 1 \\ 0 & -1 & 1 & 1 & 1 \end{bmatrix}$

Since the number of columns of A determines n, so $n = 5$ and the number of rows of A determines m, so $m = 3$. Therefore, we take

$$x = (x_1, x_2, x_3, x_4, x_5)$$

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Thus T is defined by

$$T(x) = Ax = \begin{bmatrix} 3 & 1 & 0 & 2 & 1 \\ 1 & 0 & 0 & 1 & 1 \\ 0 & -1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}$$

$$= \begin{bmatrix} 3x_1 + x_2 + 2x_4 + x_5 \\ x_1 + x_4 + x_5 \\ -x_2 + x_3 + x_4 + x_5 \end{bmatrix}$$

So $T(x_1, x_2, x_3, x_4, x_5) = (3x_1 + x_2 + 2x_4 + x_5, x_1 + x_4 + x_5, -x_2 + x_3 + x_4 + x_5)$

4. The matrix of a linear transformation $T : R^3 \to R^3$ is

$$\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & -1 \\ -1 & -1 & 0 \end{bmatrix}.$$

Find T in terms of coordinates and its matrix with respect to the basis

$v_1 = (0, 1, 2)$, $v_2 = (1, 1, 1)$, $v_3 = (1, 0, -2)$.

Sol. Let $x = (x_1, x_2, x_3)$, then we have

$$T(x) = Ax = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & -1 \\ -1 & -1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_2 + x_3 \\ x_1 - x_3 \\ -x_1 - x_2 \end{bmatrix}.$$

Thus $T(x_1, x_2, x_3) = (x_2 + x_3, x_1 - x_3, -x_1 - x_2)$

Now $T(v_1) = T(0, 1, 2) = (3, -2, -1)$

$= \alpha v_1 + \beta v_2 + \gamma v_3$

$= \alpha(0, 1, 2) + \beta(1, 1, 1) + \gamma(1, 0, -2)$

$= (\beta + \gamma, \alpha + \beta, 2\alpha + \beta - 2\gamma)$.

Hence
$$\beta + \gamma = 3 \quad (1)$$
$$\alpha + \beta = -2 \quad (2)$$
$$2\alpha + \beta - 2\gamma = -1 \quad (3)$$

Multiplying (1) by 2 and adding to (3), we get: $2\alpha + 3\beta = 5$

Multiplying equation (2) by 2, we have: $2\alpha + 2\beta = -4$

From these equations we find $\beta = 9$ and so we get $\alpha = -11$, $\gamma = -6$

Thus $T(v_1) = -11v_1 + 9v_2 - 6v_3$ (A)

Also $T(v_2) = T(1, 1, 1) = (2, 0, -2)$

$= \alpha v_1 + \beta v_2 + \gamma v_3$

$= \alpha(0, 1, 2) + \beta(1, 1, 1) + \gamma(1, 0, -2)$

$= (\beta + \gamma, \alpha + \beta, 2\alpha + \beta - 2\gamma)$

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Therefore, $(2, 0, -2) = (\beta+\gamma, \alpha+\beta, 2\alpha+\beta-2\gamma)$

$$\beta+\gamma = 2 \quad (4)$$
$$\alpha+\beta = 0 \quad (5)$$
$$2\alpha+\beta-2\gamma = -2 \quad (6)$$

Multiplying (4) by 2 and adding to (6), we get $2\alpha+3\beta = 2$

Multiplying equation (5) by 2, we have: $2\alpha+2\beta = 0$

From these equations, we find $\beta = 2$ and so we get $\alpha = -2, \gamma = 0$

Hence $T(v_2) = -2v_1 + 2v_2 + 0v_3$ (B)

Again $T(v_3) = T(1, 0, -2) = (-2, 3, -1)$

$$= \alpha v_1 + \beta v_2 + \gamma v_3$$
$$= \alpha(0, 1, 2) + \beta(1, 1, 1) + \gamma(1, 0, -2)$$
$$= (\beta+\gamma, \alpha+\beta, 2\alpha+\beta-2\gamma)$$
$$\beta+\gamma = -2 \quad (7)$$
$$\alpha+\beta = 3 \quad (8)$$
$$2\alpha+\beta-2\gamma = -1 \quad (9)$$

Multiplying (7) by 2 and adding to (8), we get $2\alpha+3\beta = -5$

Multiplying equation (8) by 2, we have: $2\alpha+2\beta = 6$

From these equations, we find $\beta = -11$ and so we get $\alpha = 14, \gamma = 9$

So $T(v_3) = 14v_1 - 11v_2 + 9v_3$ (C)

Hence the matrix of linear transformation is obtained from equations (A), (B) and (C) as

$$\begin{bmatrix} -11 & -2 & 14 \\ 9 & 2 & -11 \\ -6 & 0 & 9 \end{bmatrix}$$

5. A linear transformation $T: R^2 \to R^3$ maps the vector $(1, 1)$ into $(0, 1, 2)$ and the vector $(-1, 1)$ into $(2, 1, 0)$. What matrix does T represent with respect to the standard bases for R^2 and R^3?

Sol. $T(1, 1) = (0, 1, 2)$, $T(-1, 1) = (2, 1, 0)$.

Let $x = (x_1, x_2) \in R^2$ and $A = \begin{bmatrix} a & b \\ c & d \\ e & f \end{bmatrix}$ be the matrix of the linear transformation $T: R^2 \to R^3$. Then

$$T(x) = Ax = \begin{bmatrix} a & b \\ c & d \\ e & f \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} ax_1 + bx_2 \\ cx_1 + dx_2 \\ ex_1 + fx_2 \end{bmatrix} \quad (A)$$

i.e., $T(x_1, x_2) = (ax_1 + bx_2, cx_1 + dx_2, ex_1 + fx_2)$

Now $T(1, 1) = (a+b, c+d, e+f) = (0, 1, 2)$

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Therefore, $a+b = 0,\ c+d = 1,\ e+f = 2$ (1)

Also, we have

$$T(-1, 1) = (-a+b, -c+d, -e+f) = (2, 1, 0)$$

So $-a+b = 2, -c+d = 1, -e+f = 0$ (2)

From equations (1) and (2) we have

$$a = -1,\ b = 1,\ d = 1,\ c = 0,\ e = 1,\ f = 1$$

Hence from (A), $T(x_1, x_2) = (-x_1 + x_2, x_2, x_1 + x_2)$

Using the standard basis $\{(1, 0), (1, 0)\}$ for R^2 and the standard basis $\{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$ for R^3, we have

$$T(1, 0) = (-1, 0, 1) = -1(1, 0, 0) + 0(0, 1, 0) + 1(0, 0, 1)$$

and $T(1, 0) = (1, 1, 1) = -1(1, 0, 0) + 0(0, 1, 0) + 1(0, 0, 1)$

Thus the required matrix of linear transformation with respect to the standard bases is $\begin{bmatrix} -1 & 1 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}$.

6. The standard basis for the vector space M_{mn} consists of mn distinct matrices of order $m \times n$ each having a single 1 and all other entries equal to zero.

Let $T : M_{23} \to M_{32}$ be defined by $T(A) = A^T, A \in M_{23}$. Find the matrix of T relative to the standard bases for M_{23} and M_{32}.

Sol. Standard basis for M_{23} is

$$\left\{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right\}$$

Standard basis for M_{32} is

$$\left\{\begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}\right\}$$

$$= \{e_1, e_2, e_3, e_4, e_5, e_6\}.$$

$T : M_{23} \longrightarrow M_{32}$ is defined by $T(A) = A^T$

Thus, $T\left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$

$$= 1e_1 + 0e_2 + 0e_3 + 0e_4 + 0e_5 + 0e_6$$

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$$T\left(\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$= 0e_1 + 0e_2 + 1e_3 + 0e_4 + 0e_5 + 0e_6 .$$

$$T\left(\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \end{bmatrix}$$

$$= 0e_1 + 0e_2 + 0e_3 + 0e_4 + 1e_5 + 0e_6$$

$$T\left(\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$= 0e_1 + 1e_2 + 0e_3 + 0e_4 + 0e_5 + 0e_6$$

$$T\left(\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}\right) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{bmatrix}$$

$$= 0e_1 + 0e_2 + 0e_3 + 1e_4 + 0e_5 + 0e_6$$

$$T\left(\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= 0e_1 + 0e_2 + 0e_3 + 0e_4 + 0e_5 + 1e_6$$

Matrix of T is

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

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